

KEY TO  
ROBINSON'S  
NEW GEOMETRY & TRIGONOMETRY  
CONIC SECTIONS  
AND  
ANALYTICAL GEOMETRY.

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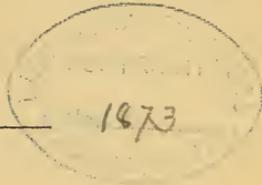
AND

CONIC SECTIONS AND ANALYTICAL GEOMETRY.

WITH

SOME ADDITIONAL ASTRONOMICAL PROBLEMS.

DESIGNED FOR TEACHERS AND STUDENTS.



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## P R E F A C E .

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A KEY to a Text-book on the Higher Mathematics, if not a new creation, is by no means a common thing. And it is a question undecided in the minds of many, whether a Key to any mathematical work is an aid, or a hindrance to the teacher.

The value of a Key depends upon the *use* to which it is applied. Class, and school-room duties so fully occupy the time of a majority of teachers, as to make a Key a great convenience, if not a necessity. And teachers of limited acquirements, as well as private students, will often find a Key of great service.

A Key should never be used to supersede investigation and labor, but to give direction to study, and make labor more effective. It should lessen the *mechanical* labor of teaching, by showing *how* to study, and *how* to teach, by giving the best forms of analysis, the best arrangement of the work, new and improved methods of operations, and often by developing methods of solu-

tion too elaborate to find a place in the Text, thus giving to those who use a Key, more extended and enlarged views of the applications of Mathematical Science.

In the preparation of the present work, great care has been taken to give full and comprehensive solutions of the examples and problems in Robinson's Plane and Solid Geometry, Plane and Spherical Trigonometry, and Analytical Geometry, and by means of cuts, more fully to explain and demonstrate the principles involved, thus making it a sort of commentary on the Text itself, and which in the hands of a good teacher may prove a valuable auxiliary in teaching.

Several pages of miscellaneous Astronomical examples have been added after Trigonometry. As soon as the New Calculus and Surveying, which are in preparation, are completed, a Key to those works will be also added to this.

JANUARY 1st., 1862.

KEY TO  
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PRACTICAL PROBLEMS.

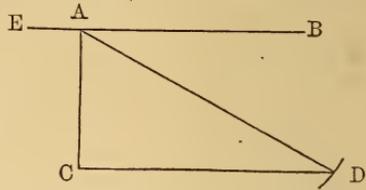
(BOOK V.—PAGE 143.)

Ex. 1. The vertical angle being  $60^\circ$ , and the sum of all the angles being 180, therefore the sum of the other two angles, is  $120^\circ$ , and, as these angles must be equal to each other, each one must be  $60^\circ$ , and the triangle is *equilateral* as well as *isosceles*. Each side is therefore equal to the base 6.

Ex. 2. As the two oblique angles of any right angled triangle equal  $90^\circ$ , and as one of them is  $30^\circ$ , the other must be  $60^\circ$ . Describe an equilateral triangle, each angle of which will be  $60^\circ$ . Divide either angle into two equal parts, by a line which will divide the opposite side into two equal parts, and the triangle into two equal triangles. The least angle, in either of these triangles is  $30^\circ$ , and the side opposite  $30^\circ$  will be half of the hypotenuse, *and this is a general truth*. Namely, *That in every right angled triangle whose least angle is  $30^\circ$ , the side opposite that angle is one-half of the hypotenuse*.

Therefore, if in this example the least side is 12, the hypotenuse is 24.

Ex. 3. Let  $AB$ ,  $CD$ , represent two parallels, and  $AC$  the perpendicular distance between them, equal 10. Take  $AD$ , in your divides, double the distance  $AC$ , and strike an arc, cutting  $CD$  in  $D$ . Join  $AD$ , and the angle  $ADC$  will be equal to  $30^\circ$ , because  $ACD$  is a right angled triangle, and its hypotenuse double the side  $AC$  (By Ex. 2). But  $DAB=ADC$ , being alternate angles, and because  $DAB=30^\circ$ , its adjacent angle is  $180^\circ - 30^\circ = 150^\circ$ .

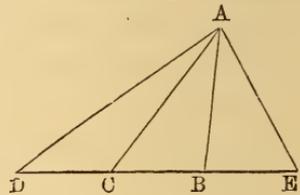


Hence the answer must be  $30^\circ$  and  $150^\circ$ .

Ex. 4. We may use the same parallels as in (Ex. 3), to illustrate this example. Let  $AC=20$ , take  $CD=20$ , and join  $AD$ , and then will  $AC=CD$ , and  $ACD$  will be an isosceles triangle, and the angles  $ADC$  and  $CAD$ , equal to each other. Their sum must be  $90^\circ$ , therefore each one of them must be  $45^\circ$ . Now as  $CAB=90^\circ$ , and  $CAD=45^\circ$ , therefore,  $DAB=45^\circ$ , and  $DAE=90^\circ + 45^\circ = 135^\circ$ .

Because  $\overline{AC}^2 + \overline{CD}^2 = \overline{AD}^2$ ,  $AC=20$ ,  
 $\overline{AD}^2 = 800 = 400 \times 2$ . Whence  $AD = 20\sqrt{2}$ . *Ans.*

Ex. 5. Let  $DE$  be one parallel, and  $AB$  a perpendicular to the other parallel. Take  $BE=BC$  and  $BD$  any distance greater than  $BC$ . Assume  $AD=15$ ,  $AC=AE=10$ , and  $AB=8$ . Now, by the well known property of the right angled triangle, we have



$CB^2 = \overline{BE}^2 - 8^2 = 6^2$ . Whence,  $CB = 6 = BE$ .

Also,  $\overline{DB}^2 = \overline{AD}^2 - 8^2 = 23 \times 7 = 161$

Whence,  $DB = \sqrt{161} = 12.69$ , and  $DE = 18.69$  } *Ans.*  
 and  $DC = 6.69$  }

Ex. 6. This problem is the same as (Ex. 5,) except in data, therefore use the same cut as before.

Make  $AB=12$ ,  $AD=20$ ,  $AC=AE=18$ .

Now,  $\overline{BD}^2 = (\overline{20}^2 - \overline{12}^2) = 32 \times 8 = 16 \times 16$ . Whence,  $BD=16$ .

$\overline{CB}^2 = \overline{BE}^2 = \overline{18}^2 - \overline{12}^2 = 30 \times 6 = 180$ .  $CB=13.416$ .

Sum,  $DE=29.416$  }  
 Diff.  $DC=2.584$  } *Ans.*

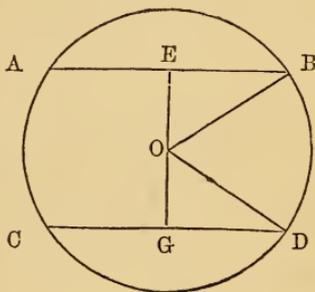
The area of the triangle  $ADE=(29.416)6=176.496$  }  
 The area of the triangle  $ADC=(2.584)6=15.504$  } *Ans.*

Ex. 7. This example produces a right angled triangle, the hypotenuse is 6, and one side 2, one half of the chord. Therefore the other side is,

$$\sqrt{6^2 - 2^2} = \sqrt{32} = 4\sqrt{2}. \quad \text{Ans.}$$

Ex. 8. The two parallel chords being equal, they must be equally distant from the center, and be on opposite sides of the center.

Let  $AB, CD$ , be those parallels ; and from the center draw  $OB, OD$  ; and  $OE, OG$ , each perpendicular to  $AB$ , and  $CD$  ; then  $EOG$ , will be one right line through the center.  $EG=6$ , therefore  $EO=3$ .  $AB=8$  ; hence,  $EB=4$  ; and in the right angled triangle  $EOB$ , we have  $\overline{OB}^2 = 4^2 + 3^2 = 25$ . Whence,  $OB=5$ . *Ans.*



Ex. 9. This is a problem of the same kind as (Ex. 8,) and therefore we may use the same cut.

Let  $AB, CD$ , be the parallel chords. Put  $AB=16$ ,

$CD=12$ , and  $EG=14$ . Place  $OB=OD=R$ ,  $EO=x$ ,  $OG=y$ . Then  $EG=x+y=14$ , or  $y=14-x$ .

$$\text{And } x^2+8^2=R^2$$

$$\text{Also } y^2+6^2=R^2$$

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$$\text{Whence } x^2+8^2=y^2+6^2=(14-x)^2+6^2$$

$$x^2+28=14^2-28x+x^2$$

$$\text{Or } 28x+28=14^2$$

$$2x+2=14, \text{ or } x=6. \text{ Whence } y=8,$$

$$\text{And } R^2=8^2+6^2=100, \text{ or } R=10. \text{ Ans.}$$

**Ex. 10.** Let  $x$  represent the perpendicular required, which is the perpendicular drawn from the vertex of an isosceles triangle to the center of its base.

$$\text{Then } x^2=15^2-5^2=20.10=2(10)^2, \text{ or } x=10\sqrt{2}. \text{ Ans.}$$

**Ex. 11.** Let  $ABC$  be the triangle, bisect  $CB$  in  $E$ ,  $AC$  in  $F$ . And join  $AE$  and  $BF$ , and through the point of intersection  $O$ , draw  $CO$ , and produce it to  $D$ .

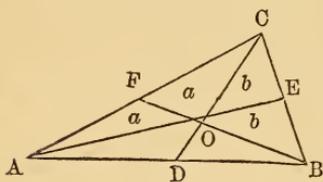
Because  $CB$  is bisected in  $E$ , the triangles  $ABE$  and  $AEC$  are equal. Or the triangle  $AEC$  is half the triangle  $ABC$ .

In like manner, we prove that the triangle  $BFC$  is half  $ABC$ , because  $AC$  is bisected in  $F$ .

But  $BOC$  consists of two equal triangles,  $BOE$  and  $EOC$ , each of which may be represented by  $b$ , and  $AOC$  consists of the equal triangles,  $AOF$  and  $FOC$ , each of which may be represented by  $a$ . Therefore  $BFC$  consists of  $2b+a$ , and  $AEC$  consists of  $2a+b$ . Whence,  $2b+a=2a+b$ .

$$\text{That is, } b=a.$$

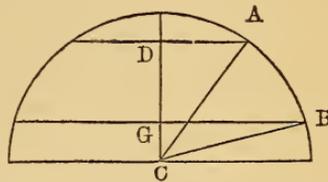
Now it is obvious that,  $BFC$  is equal to three triangles, each equal to  $a$ , or each equal to  $b$ .



Hence, the triangle  $AFB$  must be equal to  $3a$ , and  $AOB$  equal to  $2a$ . But the triangle  $ABC$  is equal to  $6a$ . That is,  $AOB$  is equal to one third of  $ABC$ .

Or,  $6a$  to  $2a$  as 3 to 1. *Ans.*

**Ex. 12.** If the diameter of a circle is 32, its radius is 16. Hence,  $CB=16$ ,  $CA=16$ , and we have two right angled triangles,  $CGB$  and  $CDA$ ;  $GB=10$ , half of one chord, and  $AD=4$ , half of the other parallel chord.



Now, these two right angled triangles give us

$$\overline{CD}^2 = \overline{16}^2 - 4^2 = 20 \times 12 = 240,$$

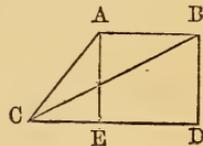
$$\overline{CG}^2 = \overline{16}^2 - \overline{10}^2 = 26 \times 6 = 156,$$

Whence  $CD = \sqrt{240} = 15.49 +$

And  $CG = \sqrt{156} = 12.49 +$

Diff. =  $DG = 3$ , if the parallels be the same side of the center, but when on opposite sides, their sum 27.98 will be the distance between the parallels.

**Ex. 13.** Here  $CD=12$ ,  $AB=8$ ,  $BD=5$ . Conceive  $CA$  and  $DB$  produced until they meet, thus forming a triangle of which  $AB$  is the base. Also conceive a line drawn from the vertex of this triangle perpendicular to the base, and designate it by  $x$ .



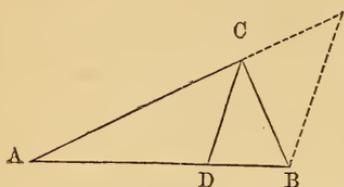
Now by proportional triangles, we shall have,

$$x : 8 :: x + 5 : 12$$

And  $12x = 8x + 40$ . Whence,  $x = 10$ .

The area of the triangle on  $AB$  as a base, is therefore

$4x=40$ . The area of the trapezoid is  $\frac{12+8}{2} \times 5=50$ , and the area of the whole triangle on the base  $CD$  is 90. *Ans.*



Ex. 14. Let  $ABC$  be the triangle  $AB=697$ ,  $AC=813$ ,  $CB=534$ .

Now by (Th. 24, B. II.), we have

$$AD : DB :: 813 : 534 \\ 271 : 178$$

Whence,  $178 AD=271 DB$ . And  $AD+DB=697$

$$178 AD+178 DB=697 \times 178$$

That is,  $271 DB+178 DB=697 \times 178$

Or  $449 DB=697 \times 178$

$$BD = \frac{697 \times 178}{449}$$

Thus we find  $BD=276.316$ , the less part.

Whence  $AD=420.684$ , the greater part. } *Ans.*

Examples 15, 16, and 17, solutions in text-book.

Ex. 18. By referring back to (Ex. 15), we observe, that  $HC$  must be taken from  $DH$ . But  $DH=\sqrt{51}$ , and  $HC=\sqrt{40}$ , hence if  $HC$  were taken from  $DH$ , the point  $C$  from  $D$  would then be  $(\sqrt{51}-\sqrt{40})$ . But  $AD=16$ . Whence, according to this condition,

$$\overline{AC}^2 = (\sqrt{51}-\sqrt{40})^2 + 16^2 \\ = 91 - 2\sqrt{2040} + 256 = 347 - 2\sqrt{2040}.$$

Or,  $\overline{AC}^2 = 347 - 2\sqrt{2040} = 347 - 90.332 = 256.668$ .

Whence,  $AC=16.021$

Area of the triangle is  $\frac{1}{2}(9\sqrt{51}+7\sqrt{40})$ . } *Ans.*

Ex. 19. Let  $E$  be the position of the observer's eye,  $EF=40$  feet,  $FH=24$  feet,  $ED$  the required distance which we designate by  $x$ , and  $DB=90$  rods, which must be reduced to feet thus  $90 \times 16\frac{1}{2}$ , or  $45 \times 33$ .



Now by proportional triangles  $EFH$ ,  $EDB$ , we have,

$$40 : 24 :: x : 45 \times 33$$

$$\text{Or, } 10 : 6 :: x : 45 \times 33 \quad \text{or, } 6x = 45 \times 330$$

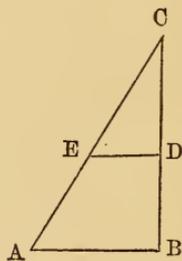
$$2x = 45 \times 110$$

$$\text{or, } x = 2475 \text{ feet. } \textit{Ans.}$$

Ex. 20. Illustrated in text-book.

Ex. 21. Let  $ABC$  be a triangle divided into two parts by the line  $DE$ , parallel to the base  $AB$ .

Let  $CDE$  be taken, as 1, and the other part  $ABDE$ , as 3. Then the whole triangle  $ABC$  will be represented by 4, the triangle  $CDE$ , being unity.



Now by (Th. 22, B. II.), we have,

$$\overline{CD}^2 : \overline{CB}^2 :: 1 : 4$$

$$\text{Square root, } CD : CB :: 1 : 2.$$

Whence  $CB=2CD$ , or  $CD$  equals one half of  $CB$ , and therefore  $CE$  equals one half of  $CA$ .

Ex. 22. Let the last triangle represent the triangle in this example also. That is, place  $AB=320$ , and as the angles adjacent to the base are  $90^\circ$  and  $60^\circ$ , the angle opposite at  $C$  must be  $30^\circ$ . The side opposite  $90^\circ$  must be

double of the side opposite  $30^\circ$ , as was explained in (Ex. 2), of this key. Therefore,  $AC=640$ .

Again, because the angle  $B=90^\circ$ .

$$\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$$

That is, 
$$\overline{320}^2 + \overline{BC}^2 = \overline{640}^2$$

Or, 
$$\overline{BC}^2 = \overline{640}^2 - \overline{320}^2 = (320)^2(2^2 - 1) = \overline{320}^2 \times 3.$$

Whence, 
$$BC = 320\sqrt{3} = 554.24. \text{ Ans.}$$

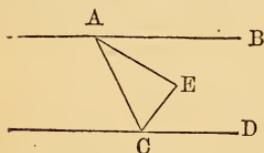
Ex. 25. Draw  $AB$  of any indefinite length, and from the point  $A$  draw  $AC$ , making an angle of  $32^\circ$  with  $AB$ , and from  $B$  draw the line  $BC$ , making with  $BA$  an angle equal to  $84^\circ$ . Produce  $AB$  to  $D$ . Because the sum of the three angles of any triangle is equal to two right angles, the angle  $C$  of the triangle  $ABC$ , must be

$$C = 180^\circ - (32^\circ + 84^\circ) = 64.$$

The exterior angle  $CBD = 180^\circ - 84^\circ = 96^\circ$ , and because  $EBD$  is one half of it, the exterior angle  $EBD$  of the triangle  $AEB$  is  $48^\circ$ . But the exterior angle  $EBD = E + EAB = E + 16^\circ$ .

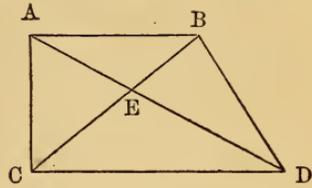
Therefore,  $E + 16^\circ = 48^\circ$ , and  $E = 32^\circ$ , one half the angle  $C$ , a general result.

Ex. 26. Let  $AB$  and  $CD$  be two parallels, and  $AC$  a line



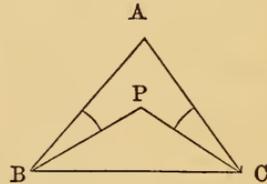
between them; because the two interior angles between two parallels are equal to two right angles, therefore, one half of each of them, added together, must make one right angle. That is, the angle  $EAC$ , and  $ECA$  together, make one right angle, therefore the third angle at  $E$ , of the triangle  $AEC$ , must be a right angle.

Ex. 27. Let  $ABCD$  be any trapezoid, and draw its diagonals  $AD$  and  $CB$ , intersecting in  $E$ . The two triangles  $AEB$  and  $CED$  are equi-angular and similar. The opposite angles at  $E$  are equal; and because,  $AB$  and  $CD$  are parallel, the alternate angles  $EAB$  and  $EDC$  are equal, and so are the alternate angles  $EBA$  and  $ECD$ , therefore the two triangles  $AEB$  and  $CED$ , are equi-angular, and  $AB$  and  $CD$ , sides opposite the equal angles  $E$ , are homologous. Therefore, by (Th. 20, B. II.), we have

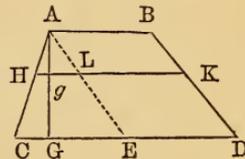


$$\overline{AB}^2 : \overline{CD}^2, \text{ as } AEB \text{ is to } CED.$$

Ex. 28. Let  $ABC$  be any triangle, the sum of its three angles is  $180^\circ$ . Take any point  $P$  within it, and draw  $PB, PC$ . If we designate the angle between  $AB$  and  $PB$  by  $a$ , and between  $AC$  and  $PC$  by  $b$ , then the angles at the base of the triangle  $PBC$ , will be less than the angles at the base of the triangle  $ABC$  by the angles  $a$  and  $b$ . But as the sum of the angles of every triangle make two right angles, therefore the angle  $BPC$  is greater than the angle  $A$ , by the sum of the two angles  $a$  and  $b$ .



Ex. 29. Let  $AB$  and  $CD$  be 12 and 20 respectively, and the perpendicular  $AG$  equal 8. Let  $HK$  represent the line between the parallels. Then  $HL = 14\frac{1}{2} - 12 = 2\frac{1}{2}$ . And we have,



$$CE : AG = HL : Ag$$

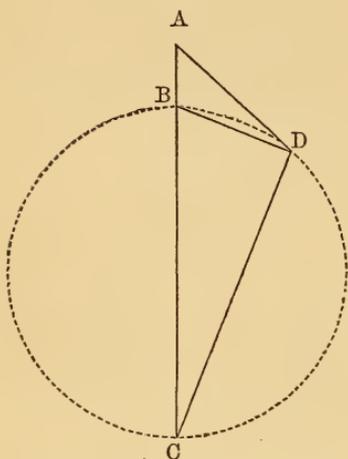
Or,  $8 : 8 = 2\frac{1}{2} : Ag.$

Hence,  $Ag=2\frac{1}{2}$ , and  $gG=5\frac{1}{2}$ .

Area of  $ABCD$  is  $\frac{20+12}{2} \times 8 = 128$ .

Area of  $ABKH$  is  $\frac{14.5+12}{2} \times 2\frac{1}{2} = 33\frac{1}{8}$ .

Area of  $HKDC$  is  $\frac{20+14.5}{2} \times 5\frac{1}{2} = 94\frac{1}{8}$ .



Ex. 30. The triangles  $ACD$  and  $ADB$  are similar, and therefore, the sides about the angle  $A$  are proportional, giving the proportion

$$AC : AD :: AD : AB.$$

But  $AC$  expressed in feet is the product of 7956 by 5280, plus  $AB=40$  feet.

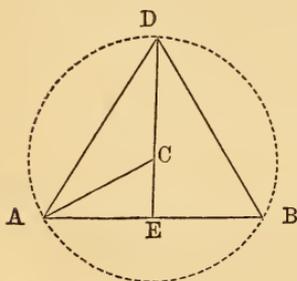
$$7956 \times 5280 = 42007680.$$

Now the above proportion becomes,

$$42007720 : AD :: AD : 40$$

Whence,  $\overline{AD}^2 = 1680308800$

Sq. root  $AD = 40992$  feet nearly, or 7 miles and 4032 feet.



Ex. 31. As the triangle  $ABD$  in the circle is to be equilateral, it must also be equiangular. Hence, the line  $AC$  from one of the angles to the center must bisect the angle  $A$ , and each division must be  $30^\circ$ .  $AC=2$ , and the side opposite

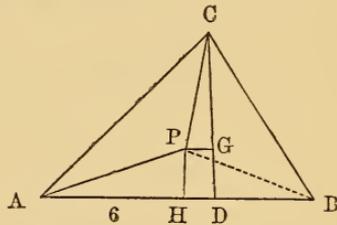
$30^\circ$  is half  $AC$ , therefore,  $CE=1$ , and  $DE=3$ . Place  $AE=x$ .

Then, the right angled triangle  $AEC$  gives  $x^2+1=4$ .

Whence,  $x=\sqrt{3}$ . But  $AE$  into  $DE$ , is the area sought.

That is,  $3\sqrt{3}$ , is the area of the triangle  $ABD$ .

Ex. 32. Let  $AB=12$ ,  $AC=11$ ,  $CB=10$ . From  $C$  draw  $CD$  perpendicular to  $AB$ . Take  $H$  the middle point between  $A$  and  $B$ , and draw  $HP$  at right angles to  $AB$ , until it meets a perpendicular drawn from the middle point between  $A$  and  $C$ ; then  $P$  will be the location of the well. Draw  $PG$  parallel to  $AB$ ,  $AH=HB=6$ .  $PH$  is common to the two triangles  $APH$  and  $PHB$ , therefore,  $AP=PB$ ,  $AP=PC$  by construction. Place  $PH=x$  and  $HD=y$ .



Then,  $AD=6+y$  and  $DB=6-y$ .

Now  $(6+y)^2+\overline{CD}^2=11^2$ , and  $(6-y)^2+\overline{CD}^2=10^2$ .

By substitution,  $24y=121-100=21$ , or  $y=\frac{7}{8}$ .

This value of  $y$  substituted in the preceding equation gives

$$(5\frac{1}{8})^2+\overline{CD}^2=100,$$

$$\text{Or, } 25+\frac{1^6}{8}+\frac{1}{64}+\overline{CD}^2=100,$$

$$\text{Or, } 210+\frac{1}{8}+8\overline{CD}^2=800.$$

Whence,  $8\overline{CD}^2=589.875$ , and  $CD=8.587$ .

Secondly. In the right angled triangles  $AHP$  and  $CPG$ , we have,

$$6^2+x^2=R^2. \quad R=AP=PC=PB.$$

$$\text{And } (8.587-x)^2+(\frac{7}{8})^2=R^2.$$

From these two equations, we obtain  $x$ , and afterwards  $R$ .

Ex. 33. The required perpendicular will obviously bisect the base of the triangle, and divide the figure into two equal right angled triangles. Represent the common perpendicular by  $x$ . Then  $x^2 + 36 = 400$ , or,  $x^2 = 364$ ,  $x = 19.07$ .

$$\text{Area sought} = (19.07)6.$$

Ex. 35. This problem is a supplement to (Ex. 34), which is explained in the text-book.

In (Ex. 34), we have the two sides of a right angled triangle (27.3) and (35.76). This problem demands the hypotenuse to that triangle, and its division into parts in the ratio of (27.3) to (35.76). (Th. 24, B. II.)

Place  $a = 27.3$ , and  $b = 35.76$ , and  $x + y =$  the required hypotenuse, then

$$x + y = \sqrt{a^2 + b^2} \quad (1)$$

But,  $x : y :: a : b$ . Or,  $bx = ay$  (2)

Multiply (1) by  $b$ , then

$$bx + by = b\sqrt{a^2 + b^2}$$

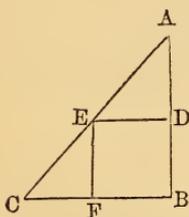
That is,  $ay + by = b\sqrt{a^2 + b^2}$

$$y = \frac{b\sqrt{a^2 + b^2}}{a + b}$$

And  $x = \frac{a\sqrt{a^2 + b^2}}{a + b}$ .

Ex. 36. If 6 feet makes a descent of 1 foot, what distance will be required for a descent of 4 feet. Represent that distance by  $x$ .

Then,  $6 : 1 :: x : 4$ . Or,  $x = 24$ . *Ans.*



Ex. 37. Let  $ABC$  represent the triangle, and  $DE$  a side of the inscribed square.

Place  $AD = x$ ,  $DE = 12 = a$ .

Now by proportional triangles we have,

$$x : a :: x + a : BC. \quad \text{Whence } BC = \frac{a(x + a)}{x}$$

Also,  $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$

That is,  $(x+a)^2 + \frac{a^2}{x^2}(x+a)^2 = \overline{35}^2$

Or,  $x^2(x+a)^2 + a^2(x+a)^2 = \overline{35}^2 x^2$

That is,  $(x^2 + a^2)(x+a)^2 = \overline{35}^2 x^2$

Or,  $(x^2 + a^2)(x^2 + 2ax + a^2) = (35)^2 x^2$

Whence  $(x^2 + a^2)^2 + 2ax(x^2 + a^2) = (35)^2 x^2$ .

Place  $y = x^2 + a^2$ . Then,  $y^2 + 2ax.y = (35)^2 x^2$ .

$$y^2 + 2ax.y + a^2 x^2 = (a^2 + (35)^2)x^2$$

$$= 1369x^2.$$

Square root  $y + ax = 37x,$

$$y = 25x.$$

But  $y = x^2 + 144$ . Therefore,  $x^2 + 144 = 25x$

$$x^2 - 25x = -144.$$

From this equation we find,  $x = 16$ , or  $9$ .

That is,

$AD = 16$  or  $9$ . But  $DB = 12$ , whence  $AB = 28$  or  $21$ . *Ans.*

To find  $BC$  we have  $BC = a \frac{(x+a)}{x} = \frac{12(16+12)}{16} = 21$ .

Or,  $\frac{12(9+12)}{9} = 28$ .

**Ex. 38.** Taking the triangle as in the preceding Example, we observe that the least hypotenuse in a triangle containing the same square, must require  $AD = DE$ , or  $AB = BC$ .

If  $AD = DE$ ,  $AB$  must be equal to  $2DE$ , because  $DB = DE$ . Therefore  $AB = 2a$ , and  $BC = 2a$ . Each side double the side of the square.

## B O O K V I I .

### P R A C T I C A L P R O B L E M S .

(Page 229.)

Ex. 1. The result sought is  $\frac{1}{6}\pi D^3$ .  $\frac{1}{6}\pi=0.5236$ , and  $D=12$ . Whence,  $\frac{1}{6}\pi D^3=(1728 \times 0.5236)=904.78$ . *Ans.*

Ex. 2. The result required in this problem, is expressed by  $\frac{1}{3}\pi H^2(3R-H)$ . (See Geometry, page 229).

When applied to this problem,  $H=3$ ,  $R=6$ ,  $\pi=3.1416$ . Whence,

$$\frac{1}{3}\pi H^2(3R-H)=(1.0472)9 \times 15=141.372 \text{ cubic in. } \textit{Ans.}$$

Ex. 3. To solve this problem, we place  $4\pi R^2=68$ ,  $2R$  being the value sought. Whence,

$$\sqrt{\frac{68}{3.1416}}=4.652. \textit{Ans.}$$

Ex. 4. In this problem the result sought is  $2\pi RH$ , but in the preceding Example, the value of  $2R$  was found to be 4.652 feet.  $H=2$  feet.  $2\pi=6.2832$ .

Therefore, the result sought is the product of the factors,  
 $(4.652)(6.2832)=29.229+$  square feet. *Ans.*

Ex. 5. For the solidity of any sphere, we have  $\frac{1}{6}\pi D^3$ ,  $\frac{1}{6}\pi=0.5236$ , and in this problem  $D=4.652$ . Whence,

$$\frac{1}{6}\pi D^3=(4.632)^3(0.5236)=52.72. \textit{Ans.}$$

For the solidity of the segment, we have the general formula,  $\frac{1}{3}\pi H^2(3R-H)$ , and for this example we have  $H=2$  feet, and  $R=2.326$ . Whence,  $3R-H=4.978$ , and the solidity sought is the product of the three factors,

$$(1.0472)4(4.978), \text{ which is } 20.852. \text{ Ans.}$$

Ex. 6. The general expression for the solidity of a segment of a sphere having two bases, is

$$\frac{1}{6}\pi H^2 + \frac{\pi H}{2}(R'^2 + R''^2)$$

$R'$  and  $R''$  being the radii of the bases, and  $H$  the perpendicular distance between them.

In this Example the radius of the sphere is 10, and  $H=2$ .  $(R')^2$  and  $(R'')^2$  must be computed.

For the greater segment, we have,

$$(R')^2 = 100 - 9, \text{ and } (R'')^2 = 100 - 25.$$

Whence,

$$(0.5236)(8) + (3.1416)(166) = 525.7 \text{ cubic feet. Ans.}$$

For the smaller segments, we have,  $(R')^2 = 100 - 25 = 75$ .

$$\text{and } (R'')^2 = 100 - 49 = 51.$$

Whence,

$$4.1888 + (3.1416)(126) = 400.0304 \text{ cubic feet. Ans.}$$

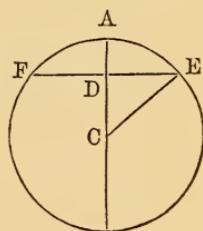
Ex. 7. Let the circle  $AFB$  represent a central section of the sphere,  $FE$  the diameter of the segment required. Let  $DE=8$ ,  $AD=4$ , and  $AC=EC=R$ ; then in the triangle  $CDE$ ,

$$DC=R-4,$$

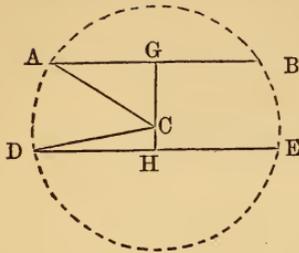
And,  $(R-4)^2 + 8^2 = R^2$ . Whence,  $R=10$ .

Now for the solidity of the segment, we have

$$\frac{1}{3}\pi H^2(3R-H) = (1.0472)16(30-4) = 435.6352. \text{ Ans.}$$



Ex. 8. Let  $AB$  and  $DE$  be the diameter of the segment, and they are parallel, and their distance asunder being 9 inches,  $AG=7$ , and  $DH=9$ , and  $AC$  and  $DC$ = each to  $R$ . Place  $CH=x$ , and the two right angled triangles  $AGC$  and  $DCH$ , give



$$7^2 + (9-x)^2 = R^2, \quad 9^2 + x^2 = R^2.$$

Whence,  $49 + 9^2 - 18x + x^2 = 9^2 + x^2$ , or  $18x = 49$ ,  $x = 2\frac{1}{2}$ .

$$\text{Now } R^2 = 81 + \left(\frac{49}{18}\right)^2 = \frac{26244 + 2401}{324} = \frac{28645}{324}.$$

$$R = 9.4027. \quad \text{Ans.}$$

For the value of the segment, we have,

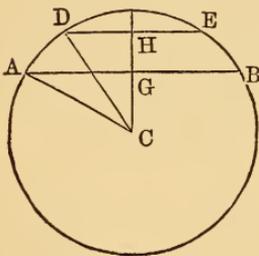
$$.5236(.729) + \frac{9.(3.1416)}{2}(81 + 49) = 2219.5 +$$

Ex. 9. In this example, the height of the segment is 6, and the solution is

$$.5236(216) + \frac{6.(3.1416)}{2}(300 + 144) = 4297.7.$$

Ex. 10. The contents of the segments will be

$$\frac{1}{6}\pi(2^3) + \frac{2\pi}{2}(100 + 36) = 431.45 \text{ cu. mi.}$$



To find the diameter,

Let  $CA = x$ ,  $CG = y$

Then,  $CH = y + 2$ . Hence,

$$x^2 = y^2 + 100 \quad (1)$$

$$x^2 = (y + 2)^2 + 36 \quad (2)$$

Whence we have

$$(y + 2)^2 = y^2 + 100$$

$$y = 15.$$

And from (1),

$$x = \sqrt{225 + 100} = 18.027, \text{ radius.}$$

# B O O K V I I I .

## APPLICATION OF ALGEBRA TO GEOMETRY.

### P R O B L E M V I .

*In a right angled triangle, having given the base, and the sum of the perpendicular and hypotenuse to find these two sides.*

Let  $ABC$  be the triangle,  $AB=b$ , and  $AC+CB=S$ , and  $BC=x$ .

Whence,  $AC=S-x$ .

But  $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$

That is,  $b^2 + x^2 = (S-x)^2 = S^2 - 2Sx + x^2$

$$2Sx = S^2 - b^2$$

$$x = \frac{S^2 - b^2}{2S} = CB, \quad AC = S - \frac{S^2 - b^2}{2S} = \frac{S^2 + b^2}{2S}.$$



### P R O B L E M V I I .

*Given the base and altitude of a triangle, to divide it into three equal parts, by lines parallel to the base.*

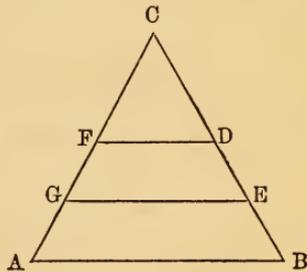
Let  $ABC$  represent the  $\triangle$ . Conceive a perpendicular let drop from  $C$  to the base  $AB$ , and represent it by  $b$ . Put  $2a=AB$ . Then  $ab$  = the area of the triangle.

Let  $x$  be the distance from  $C$  to  $FD$ ; then by (Th. 22, Bk. II.),

$$x^2 : b^2 :: \frac{1}{3}ab : ab$$

Whence,  $x : b :: 1 : \sqrt{3}$ .

Or,  $x = \frac{b}{\sqrt{3}}$ .



If  $x$  represents the distance from  $C$  to  $GE$ , then

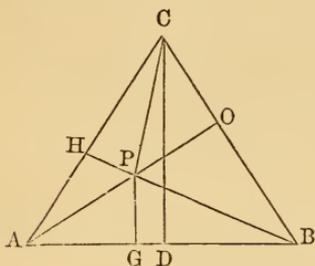
$$x^2 : b^2 :: \frac{2}{3}ab : ab.$$

$$\text{Or, } x : b :: \sqrt{2} : \sqrt{3}. \quad x = \frac{\sqrt{2} \cdot b}{\sqrt{3}}.$$

We perceive by this that the divisions of the perpendicular are independent of the base, and that we may divide the triangle into any required number of parts,  $m$ ,  $n$ ,  $p$ , etc., equal or unequal.

#### PROBLEM VIII.

*In any equilateral triangle, given the lengths of three perpendiculars drawn from any point within to the sides, to determine the sides.*



Let  $ABC$  be an equilateral triangle, and because  $CD$  is drawn perpendicular to the base, it bisects the base. Place  $AD = x$ . Then  $AB$ ,  $BC$ ,  $AC$  each equal  $2x$ . Take  $PO = a$ ,  $PG = b$ ,  $PH = c$ . The area of the triangle  $ABC$  equals  $(CD)x$ . But in the right angled triangle  $ADC$ , we have,  $\overline{CD}^2 + x^2 = 4x^2$ . Whence,  $CD = x\sqrt{3}$ , and the area of the triangle  $ABC$ , therefore is  $x^2\sqrt{3}$ .

Again, the area of the triangle  $APB$ , is  $\left(PG \times \frac{AB}{2}\right)$ , or  $bx$ . Area  $BPC = ax$  and  $APC = cx$ , and the sum of these three interior triangles, equals the triangle  $ABC$ .

$$\text{That is, } x^2\sqrt{3} = ax + bx + cx$$

$$\text{Whence, } 2x = \frac{2(a+b+c)}{\sqrt{3}} \text{ the length of a side.}$$

## PROBLEM IX.

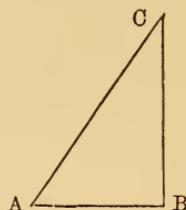
In a right angled triangle, having given the base (3), and the difference between the hypotenuse and perpendicular (1) to find the sides.

Place  $AB=3$ ,  $AC-CB=1$ .

Make  $CB=x$ , then  $AC=1+x$ .

And  $9+x^2=1+2x+x^2$ .

Whence,  $2x=8$ ,  $x=4$ . Then,  $AC=5$



## PROBLEM X.

In a right angled triangle, having given then hypotenuse (5), and the difference between the base and perpendicular (1), to find the sides.

Place  $AC=5$ ,  $CB-AB=1$ ,  $AB=x$ .

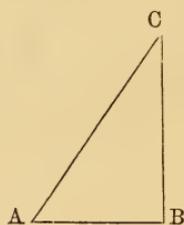
Then  $CB=1+x$ .

Now  $x^2+(1+x)^2=25$ .

$2x^2+2x=24$ .

Whence,  $x=3$ ,  $BC=4$ .

Ans.

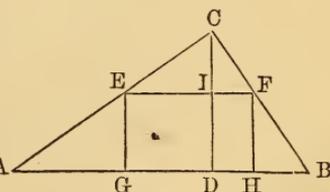


## PROBLEM XI.

Having given the area of a rectangle inscribed in a given triangle to determine the side of the rectangle.

When we say that a triangle is given, we mean that the base and perpendicular are given.

Let  $ABC$  be the triangle,  $AB=b$ ,  $CD=p$ ,  $CI=x$ ; then  $ID=p-x$ .



By proportional triangles, we have,

$$CI : EF :: CD : AB$$

That is,  $x : EF :: p : b$ .  $EF = \frac{bx}{p}$ .

By the problem  $\frac{bx}{p}(p-x) = a$ . The symbol  $a$  being the given area.

Whence,  $x^2 - px = -\frac{ap}{b}$ .  $x = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - \frac{ap}{b}}$ .

#### PROBLEM XII.

*In a triangle having given the ratio of the two sides, together with both the segments of the base, made by a perpendicular from the vertical angle, to determine the sides of the triangle.*

Let  $ACB$  be the  $\triangle$ , (see last figure.)  $AD = a$ ,  $BD = b$ , and  $CD = x$ . Then  $AC = \sqrt{a^2 + x^2}$ , and  $CB = \sqrt{b^2 + x^2}$ .

The ratio of  $AC$  to  $CB$  is given, and let that ratio be as 1 to  $r$ ; then

$$\sqrt{a^2 + x^2} : \sqrt{b^2 + x^2} :: 1 : r.$$

Whence,  $a^2 + x^2 : b^2 + x^2 :: 1 : r^2$ .

Or,  $b^2 + x^2 = a^2 r^2 + r^2 x^2$ .

Or,  $x^2 = \frac{a^2 r^2 - b^2}{1 - r^2}$ .

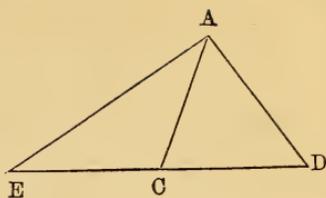
But  $AC = \sqrt{a^2 + x^2}$ , and as  $x^2$  is now known,  $AC$  is known.

#### PROBLEM XIII.

*In any triangle having given the base, the sum of the other two sides, and the length of a line drawn from the vertical angle to the middle of the base, to find the sides of the triangle.*

Let  $ADE$  be the  $\triangle$ . Suppose  $C$  to be the middle of the base.

Put  $AC=a$ ,  $DC$  or  $CE=b$ ,  $AE=x$ ,  $DA+AE=c$ ; then  $DA=c-x$ .



Now by (Th. 42, B. I.), we have,

$$(DA)^2 + (AE)^2 = 2(AC)^2 + 2(DC)^2$$

That is,  $c^2 - 2cx + 2x^2 = 2a^2 + 2b^2$ .

Or,  $4x^2 - 4cx + c^2 = 4a^2 + 4b^2 - c^2$ .

$$2x - c = \sqrt{4a^2 + 4b^2 - c^2}.$$

Whence  $x$  becomes known, and consequently the sides become known.

#### PROBLEM XIV.

To determine a right angled triangle, having given the length of two lines drawn from the acute angles to the middle of the opposite sides.

Let  $ABC$  be the triangle. Bisect  $AB$  in  $E$ ,  $CB$  in  $D$ , and join  $CE=a$ , and  $AD=b$ .

Place  $DB=x$ , and  $BE=y$ .

Now in the two right angled triangles  $ABD$ , and  $CBE$ , we have,

$$4y^2 + x^2 = b^2 \quad (1)$$

$$4x^2 + y^2 = a^2 \quad (2)$$

Sum  $5(x^2 + y^2) = (a^2 + b^2)$

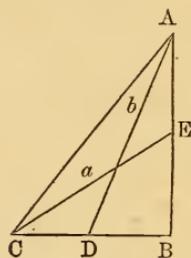
Or,  $x^2 + y^2 = \frac{1}{5}(a^2 + b^2) \quad (3)$

But  $4x^2 + y^2 = a^2 \quad (4)$

Sub. (3) from (4), and  $3x^2 = \frac{4}{5}a^2 - \frac{1}{5}b^2$ .

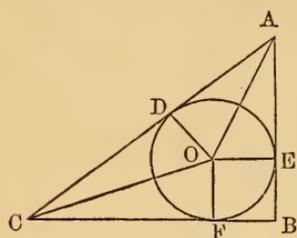
Whence,  $x = \left( \frac{4}{15}a^2 - \frac{1}{15}b^2 \right)^{\frac{1}{2}}$

Also,  $y = \left( \frac{4}{15}b^2 - \frac{1}{15}a^2 \right)^{\frac{1}{2}}$ .



## PROBLEM XV.

To determine a right angled triangle having given the perimeter and the radius of the inscribed circle.



Let  $ABC$  be the triangle, draw  $AO, CO$  to the centre of the circle, and it is obvious that the two right angled triangles  $AEO, ADO$ , are equal to each other. Also,  $CFO = CDO$ , and  $BEOF$  is a square, each side equal to the radius of the circle.

Place  $EB = BF = r$ ,  $AE = AD = x$ , and  $CF = CD = y$ . Designate the given perimeter by  $2p$ .

$$\text{Then} \quad 2x + 2y + 2r = 2p$$

$$\text{Or,} \quad x + y + r = p$$

$$\text{Or,} \quad x + y = (p - r) \quad (1)$$

Also, by the right angled triangle, we find,

$$(x + r)^2 + (y + r)^2 = (x + y)^2 \quad (2)$$

$$\text{Or,} \quad x^2 + 2rx + r^2 + y^2 + 2ry + r^2 = x^2 + 2xy + y^2.$$

$$\text{Reducing and,} \quad 2r(x + y) + 2r^2 = 2xy.$$

$$\text{Or,} \quad r(p - r) + r^2 = xy.$$

$$\text{Whence,} \quad xy = rp \quad (3)$$

Equations (1) and (3) will readily give the values of  $x$  and  $y$ , which solves the problem.

## PROBLEM XVI.

To determine a triangle, having given the base, the perpendicular, and the ratio of the two sides.

Let  $ABC$  be the  $\triangle$ .  $AB=b$ ,  
 $CD=a$ ,  $DB=x$ . Then

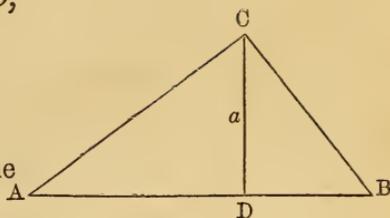
$$AC = \sqrt{(b-x)^2 + a^2}.$$

$$CB = \sqrt{a^2 + x^2}.$$

Let the given ratio of the  
 sides be as  $m$  to  $n$ ; then

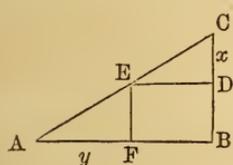
$$\sqrt{(b-x)^2 + a^2} : \sqrt{a^2 + x^2} :: m : n.$$

This proportion will give the value of  $x$ , then  $AC$  and  $CB$   
 will be known.



#### PROBLEM XVII.

To determine a right angled triangle, having given the  
 hypotenuse, and a side of its inscribed square.



Let  $ABC$  be the triangle, and  $DE=a$ ,  
 a side of the inscribed square. Place  
 $CD=x$ , and  $AF=y$ .

Also, let  $AC=h$ .

$$\text{Then, } (x+a)^2 + (y+a)^2 = h^2. \quad (1)$$

$$\text{And } x : a :: a : y, \quad (2)$$

Expanding (1), transposing, etc., gives

$$x^2 + y^2 + 2a(x+y) = h^2 - 2a^2. \quad (3)$$

$$\text{From (2) } 2xy = 2a^2. \quad (4)$$

$$\text{Sum of (3) and (4) } x^2 + 2xy + y^2 + 2a(x+y) = h^2. \quad (5)$$

$$\text{Or, } (x+y)^2 + 2a(x+y) = h^2. \quad (6)$$

$$\text{Whence, } x+y = -a \pm \sqrt{h^2 + a^2}. \quad (7)$$

Equations (4) and (7) will give the value of  $x$  and  $y$ , and  
 solve the problem.

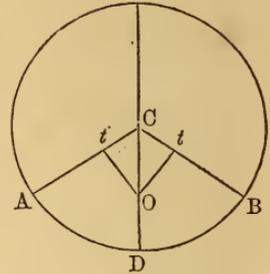
#### PROBLEM XVIII.

To determine the radii of three equal circles inscribed in  
 a given circle, touching each other, and each touching the  
 circumference of the given circle.

Every circle consists of  $360^\circ$ , and therefore if three circles

are to be placed in a given circle, each one of them must occupy a section of  $120^\circ$ , one third of  $360^\circ$ .

Let  $ACB$  be one of those sectors, and it is obvious that the centre of one of the interior circles must be on the radius  $CD$  of the larger circle,  $CD$  dividing the angle  $ACB$  into two equal parts.



It is also obvious that the interior circle which is placed in the sector  $ACB$ , must touch the given circle in the point  $D$ , and touch the radii  $CA$ ,  $CB$ , in the points  $t$ ,  $t$ , equidistant from  $C$ , and so situated that perpendiculars  $Ot$ ,  $Ot$ , shall be equal to  $OD$ .

Place  $OD = Ot = x$ , then  $x$  represents the radius of one of the required circles, and let  $R$  represent  $CD$ , the radius of the given circle.

In the right angled triangles  $OtC$  the angle  $tCO = 60^\circ$ , angle  $t = 90^\circ$ , therefore the angle  $tOC = 30^\circ$ ,  $CO = R - x$ ; and in every right angled triangle, when one of the acute angles is  $30^\circ$ , the side opposite to that angle is one half the hypotenuse, therefore,  $tC = \frac{R-x}{2}$

$$\text{But} \quad \overline{tO}^2 + \overline{tC}^2 = \overline{CO}^2.$$

$$\text{That is,} \quad x^2 + \frac{(R-x)^2}{4} = (R-x)^2.$$

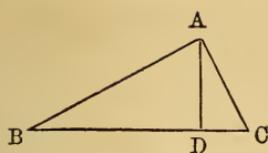
$$\text{Or,} \quad x^2 = \frac{3}{4}(R^2 - 2Rx + x^2).$$

$$x^2 = 3R^2 - 6Rx.$$

$$x = R(2\sqrt{3} - 3). \quad \text{Ans.}$$

#### PROBLEM XIX.

*In a right angled triangle, having given the perimeter, and the perpendicular let fall from the right angle on the hypotenuse, to determine the triangle; that is, its sides.*



Let  $ABC$  be the  $\triangle$ , and represent its perimeter by  $p$ . Put  $AD=a$ ,  $AB=x$ ,  $AC=y$ . Then  $BC=p-x-y$ .

Because  $BAC$  is a right angle,

$$x^2 + y^2 = p^2 - 2p(x+y) + x^2 + 2xy + y^2 \quad (1)$$

$BC \cdot AD = 2$  times the area of the triangle  $ABC$ .

Also,  $AC \cdot AB = 2$  times the area of the triangle  $ABC$ .

Therefore,  $a(p-x-y) = xy$ . (2)

Reducing (1),  $2p(x+y) = p^2 + 2xy$  (3)

Double (2),  $2ap - 2a(x+y) = 2xy$  (4)

By subtraction,  $(2a+2p)(x+y) - 2ap = p^2$  (5)

Whence,  $x+y = \frac{p^2 + 2ap}{2a+2p}$  (6)

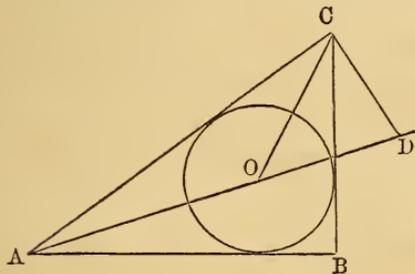
Because  $BC = p - x - y$ ,  $BC = p - \frac{p^2 + 2ap}{2a+2p} = \frac{p^2}{2a+2p}$

From (2) we observe that  $xy = \frac{ap^2}{2a+2p}$  (7)

Equations (6) and (7), will readily give  $x$  and  $y$ .

#### PROBLEM XX.

*To determine a right angled triangle, having given the hypotenuse, and the difference of two lines drawn from the two acute angles to the centre of the inscribed circle.*



Let  $ABC$  be the triangle, and  $O$  the center of its inscribed circle,  $AO$  and  $CO$  being joined, the triangle  $AOC$  is formed, and  $AO$  being produced to meet a perpendicular from  $C$ ,  $COD$  is an exterior angle.

$AO$  bisects the angle at  $A$ , and  $CO$  bisects the angle  $ACB$ .

$AO$  bisects the angle at  $A$ , and  $CO$  bisects the angle  $ACB$ .

Therefore, the angle  $COD$ , is equal to half the angles  $A$  and  $C$  of the right angled triangle  $ABC$ . Consequently  $COD$  is  $45^\circ$ ;  $D$  being  $90^\circ$ , the angle  $OCD=45^\circ$ , whence  $OD=CD$ . Let  $CO=x$ , and  $AO=d+x$ ,  $d$  being the given difference between  $AO$ ,  $OC$ . Also, let  $OD=y$ , and  $AC=h$ .

$$\text{Then} \quad 2y^2 = x^2 \quad (1)$$

$$\overline{AD}^2 + \overline{DC}^2 = \overline{AC}^2.$$

$$\text{That is,} \quad (x+y+d)^2 + y^2 = h^2 \quad (2)$$

Expanding  $x^2 + y^2 + d^2 + 2xy + 2dx + 2dy + y^2 = h^2$ .

Substituting the value of  $2y^2$ , as found in (1), and

$$2x^2 + 2xy + 2d(x+y) = h^2 - d^2$$

$$2x^2 + \frac{2x^2}{\sqrt{2}} + 2d\left(x + \frac{x}{\sqrt{2}}\right) = h^2$$

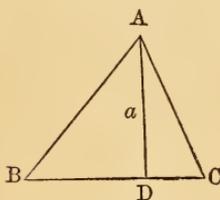
$$(2 + \sqrt{2})x^2 + (2 + \sqrt{2})dx = h^2 - d^2$$

$$x^2 + dx = \frac{h^2 - d^2}{2 + \sqrt{2}} = m.$$

$$\text{Whence,} \quad x = -\frac{d}{2} \pm \sqrt{m + \frac{d^2}{4}}.$$

#### PROBLEM XXI.

To determine a triangle, having given the base, the perpendicular and the difference of the two sides.



Let  $ABC$  be the  $\triangle$ . Put  $BD=x$ ,  $DC=y$ ,  $AC=z$ ,  $AB=z+d$ ,  $AD=a$ ,  $BC=b$ . By the conditions,

$$x+y=b \quad (1)$$

$$x^2 + a^2 = z^2 + 2dz + d^2 \quad (2)$$

$$y^2 + a^2 = z^2 \quad (3)$$

$$\overline{x^2 - y^2 = 2dz + d^2} \quad (4)$$

By subtracting,

$$\text{Factoring,} \quad (x+y)(x-y) = d(2z+d)$$

$$\text{That is,} \quad b(x-y) = d(2z+d).$$

From this we have the proportion,

$$b : (2z+d) :: d : (x-y)$$

This proportion is the following rule given in trigonometry, viz. :

*In any plane triangle, as the base is to the sum of the sides, so is the difference of the sides to the difference of the segments of the base.*

We return to the solution. From (1) we have,

$$x = b - y, \text{ whence } x^2 - y^2 = b^2 - 2by.$$

From (3)  $z = \sqrt{y^2 + a^2}$ . These values put in (4) give

$$b^2 - 2by = 2d\sqrt{y^2 + a^2} + d^2$$

$$(b^2 - d^2) - 2by = 2d\sqrt{y^2 + a^2}$$

Squaring,  $(b^2 - d^2)^2 - 4b(b^2 - d^2)y + 4b^2y^2 = 4d^2y^2 + 4a^2d^2$

Or,  $(b^2 - d^2)^2 - 4b(b^2 - d^2)y + 4(b^2 - d^2)y^2 = 4a^2d^2$

$$(b^2 - d^2) - 4by + 4y^2 = \frac{4a^2d^2}{b^2 - d^2}$$

$$b^2 - 4by + 4y^2 = \frac{4a^2d^2}{b^2 - d^2} + d^2 = m^2$$

Whence,

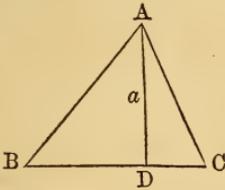
$$b - 2y = \pm m$$

And,

$$y = b + m, \text{ or } b - m.$$

#### PROBLEM XXII.

*To determine a triangle, having given the base, the perpendicular, and the rectangle, or product of the two sides.*



Let  $ABC$  be the  $\triangle$ . Put  $BD = x$ ,  $DC = y$ ,  $BC = b$ ,  $AD = a$ , and the rectangle,  $(AB)(AC) = c$ .

Now in the right angled triangles,  $ADB$ ,  $ADC$ , we have,

$$AB = \sqrt{x^2 + a^2}. \quad AC = \sqrt{y^2 + a^2}.$$

$$\text{Whence, } (\sqrt{x^2+a^2})(\sqrt{y^2+a^2})=c \quad (1)$$

$$\text{And, } x+y=b \quad (2)$$

$$\text{From (1), } x^2y^2+a^2(x^2+y^2)+a^4=c^2 \quad (3)$$

$$\text{From (2), } x^2+y^2=b^2-2xy \quad (4)$$

This value substituted in (3), gives

$$\begin{aligned} x^2y^2+a^2b^2-2a^2xy+a^4 &= c^2 \\ x^2y^2-2a^2xy+a^4 &= c^2-a^2b^2 \\ xy-a^2 &= \pm\sqrt{c^2-a^2b^2} \\ xy &= a^2 \pm\sqrt{c^2-a^2b^2} \quad (5) \end{aligned}$$

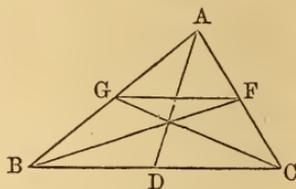
Now equations (2) and (5) will give the values of  $x$  and  $y$ .

#### PROBLEM XXIII.

To determine a triangle, having given the lengths of the three lines drawn from the three angles to the middle of the opposite sides.

Let  $ABC$  be the  $\triangle$ . Place  $AD=a$ ,  $BF=b$ , and  $CG=c$ .

Also, put  $BD=x$ , one half of the base,  $BG=y$ , one half of the side  $AB$ , and  $FC=z$ , one half of the side  $AC$ . Now by (Th. XLII., B. I.), we have,



$$\overline{AB}^2 + \overline{AC}^2 = 2\overline{BD}^2 + 2\overline{AD}^2.$$

$$\text{That is } 4y^2 + 4z^2 = 2x^2 + 2a^2 \quad (1)$$

$$\text{Similarly, } 4x^2 + 4y^2 = 2z^2 + 2b^2 \quad (2)$$

$$\text{And, } 4x^2 + 4z^2 = 2y^2 + 2c^2 \quad (3)$$

$$\text{Reduced sum, } 6x^2 + 6y^2 + 6z^2 = 2(a^2 + b^2 + c^2)$$

$$\text{Or, } 2x^2 + 2y^2 + 2z^2 = \frac{2}{3}(a^2 + b^2 + c^2) = m$$

$$\text{From (1) } 2y^2 + 2z^2 = x^2 + a^2$$

$$\text{By subtraction } 2x^2 = -x^2 + m - a^2$$

$$\text{Or, } 3x^2 = m - a^2. \quad \text{Whence, } x = \frac{\sqrt{m-a^2}}{\sqrt{3}}$$

In like manner we find  $y = \frac{\sqrt{m-c^2}}{\sqrt{3}}$ .

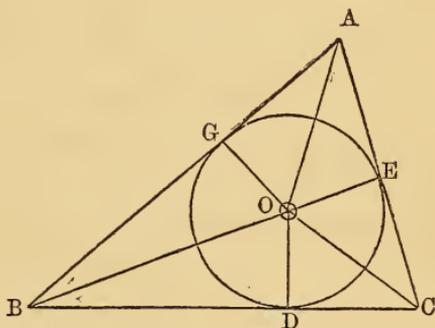
And,  $z = \frac{\sqrt{m-b^2}}{\sqrt{3}}$ . Double of these values will be the sides required.

PROBLEM XXIV.

*In a triangle having given the three sides, to find the radius of the inscribed circle.*

Let  $ABC$  be the  $\triangle$ .  
From the center of the circle  $O$ , let fall the perpendiculars  $OG$ ,  $OE$ ,  $OD$ , on the sides.

These perpendiculars are all equal, and each equal to the radius required.



Let the side opposite to the angle  $A$ , be represented by  $a$ , the side opposite  $B$ , by  $b$ , and opposite  $C$ , by  $c$ . Put  $OE$ ,  $OD$ , etc., equal to  $r$ .

It is obvious that the double area of the  $\triangle BOC$  is expressed by  $ar$ ; the double area of  $AOB$  by  $cr$ ; the double area of  $AOC$  by  $br$ . Therefore, the double area of  $ABC$  is

$$(a+b+c)r.$$

From  $A$  let drop a perpendicular on  $BC$ , and call it  $x$ .

Then  $ax =$  the double area of  $ABC$ . Consequently,

$$(a+b+c)r = ax \quad (1)$$

The perpendicular from  $A$  will divide the base  $BC$  into two

segments, one of which is  $\sqrt{c^2-x^2}$ , the other,  $\sqrt{b^2-x^2}$ , and the sum of these is  $a$ ; therefore,

$$\sqrt{c^2-x^2} + \sqrt{b^2-x^2} = a \quad (2)$$

$$\sqrt{c^2-x^2} = a - \sqrt{b^2-x^2}$$

$$c^2-x^2 = a^2 - 2a\sqrt{b^2-x^2} + b^2-x^2$$

$$2a\sqrt{b^2-x^2} = a^2 + b^2 - c^2$$

$$\sqrt{b^2-x^2} = \frac{a^2 + b^2 - c^2}{2a} = m$$

Whence,

$$b^2-x^2 = m^2$$

Or,

$$x = \sqrt{b^2-m^2}$$

This value of  $x$  put in (1), gives

$$(a+b+c)r = a\sqrt{b^2-m^2}$$

Whence,

$$r = \frac{a\sqrt{b^2-m^2}}{a+b+c},$$

the required result.

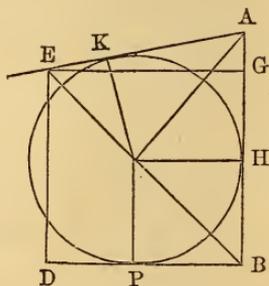
#### PROBLEM XXV.

*To determine a right angled triangle, having given the side of the inscribed square, and the radius of the inscribed circle.*

Let  $AB$  be one side of the triangle, and  $BG$  a side of the inscribed square, which we designate by  $a$ . And let  $BH=r$ , a radius of the inscribed circle.

Then  $GH = a - r = d$ .

Join  $AE$ , touching the circle at  $K$ , and  $BD$ , touching the circle at  $P$ .



Conceive the lines  $AE$  and  $BD$  produced, meeting at a point  $C$ , then  $ABC$  is the triangle sought.

Let  $AH=AK=x$ , and  $KC=CP=y$ .

Then  $AC$  = the hypotenuse  $= (x+y)$ . (Prob. XV.)

$AB=x+r$ , and  $CB=y+r$ .

$$(x+r)^2 + (y+r)^2 = (x+y)^2 \quad (1)$$

Again  $AG=x-d$ ,  $GE=a$ , and  $CD=y-d$ .

Now,  $AG : GE :: ED : CD$

That is,  $(x-d) : a :: a : y-d$ .

Whence,  $xy-d(x+y)+d^2=a^2$  (2)

Expanding and reducing (1), we obtain,

$$2r(x+y)+2r^2=2xy$$

Or,  $xy=r(x+y)+r^2$  (3)

From (2)  $xy=d(x+y)+a^2-d^2$

By subtraction  $0=(r-d)(x+y)+r^2+d^2-a^2$

Whence,  $x+y=\frac{r^2+d^2-a^2}{d-r}$ . (5)

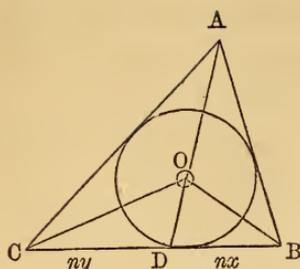
This value of  $(x+y)$  placed in equation (3) and reduced, and we obtain,

$$xy=\frac{r(d^2-a^2+dr)}{d-r}. \quad (6)$$

Equations (5) and (6) will give separate values of  $x$  and  $y$ , and thus solve the problem.

#### PROBLEM XXVI.

*To determine a triangle and the radius of the inscribed circle, having given the lengths of three lines drawn from the three angles to the center of that circle.*



Let  $ABC$  be the  $\triangle$ ,  $O$  the center of the circle.

Put  $AO=a$ ,  $OB=b$ ,  $OC=c$ .  
 $AO$  bisects the angle  $A$ .

Produce  $AO$  to  $D$ . Then because the angle  $A$  is bisected,  
 $CD : DB :: AC : AB$ .

Put  $AB=x$ ,  $AC=y$ , and let the ratio of  $AB$  to  $BD$  be  $n$ ; then  $nx=BD$  and  $ny=CD$ .

Now as the angle  $C$  is bisected by  $CO$ , we have,

$$AC : CD :: AO : OD$$

That is,  $y : ny :: a : OD$

Whence,  $OD=na$ .

Because  $AD$  bisects the angle  $A$ , we have, (Th. XX. Book III.)

$$xy=a^2(1+n)^2+n^2xy \quad (1)$$

Also,  $nx^2=b^2+na^2 \quad (2)$

And,  $ny^2=c^2+na^2 \quad (3)$

From (1),  $xy=\frac{a^2(1+n)^2}{1-n^2}=\frac{a^2(1+n)}{1-n} \quad (4)$

The product of (2) and (3), gives

$$n^2x^2y^2=(c^2+na^2)(b^2+na^2) \quad (5)$$

Squaring (4), and multiplying the result by  $n^2$ , also gives

$$n^2x^2y^2=\frac{a^4(1+n)^2n^2}{(1-n)^2} \quad (6)$$

Equating (5) and (6), gives

$$(c^2+na^2)(b^2+na^2)(1-n)^2=a^4(1+n^2)n^2$$

This equation contains only one unknown quantity  $n$ , but it rises to the fourth power—hence this problem is not susceptible of a solution under this notation, short of an equation of the 4th degree.

When  $a$ ,  $b$ , and  $c$ , are numerically given, cases may occur in which the resulting equation may be of a low degree. When  $b=c$ , then  $x=y$ .

The three sides being determined, the radius of the inscribed circle is then found by Problem XXIV.

PROBLEM XXVII.

*To determine a right angled triangle, having given the hypotenuse, and the radius of the inscribed circle.*

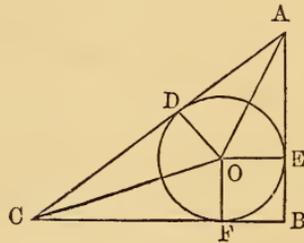
Let  $ABC$  be the  $\triangle$ . Place  $AE=x$ ,  $EB=EO=r$ , and  $CF=y$ .

Then,  $AB=x+r$

$BC=y+r$

And,  $AC=(x+y)=h$ .

Now,  $(x+r)^2+(y+r)^2=(x+y)^2$ .



Expanding and reducing the above, we have

$$r(x+y) + r^2 = xy$$

That is,  $xy = r^2 + hr$

And,  $x+y = h$ .

Whence,  $x = \frac{1}{2}h + \frac{1}{2}\sqrt{h^2 - 4hr - 4r^2}$

And,  $y = \frac{1}{2}h - \frac{1}{2}\sqrt{h^2 - 4hr - 4r^2}$ .

PROBLEM XXVIII.

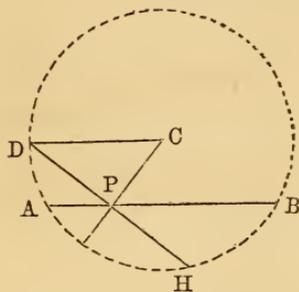
Here the problem is given in general terms—this same problem is numerically given in (Book V. Prob. 8), and is solved in this key, on page (7).

## PROBLEM XXIX.

This problem involves the same relations as (Prob. 12, Book V.), which is solved in this key, on page (9).

## PROBLEM XXX.

*The radius of a circle being given, also the rectangle of the segments of a chord, to determine the distance from the center to the point at which the chord is divided.*



Let  $C$  be the center of the circle,  $AB$  the chord, and  $P$  the given point in it. Join  $CP$  and denote that distance by  $x$ .

Through  $P$  draw the chord  $DH$  at right angles to  $PC$ . Then because  $CP$  is a line from the center perpendicular to the chord  $DH$ , it must bisect that chord. (Th. I. Book III.)

$DPC$  is a right angled triangle,  $DC=r$ , then

$$DP = \sqrt{r^2 - x^2}.$$

But because  $DH$ , and  $AB$  intersect at  $P$ , we have,

$$AP.PB = DP.PH = \overline{DP}^2$$

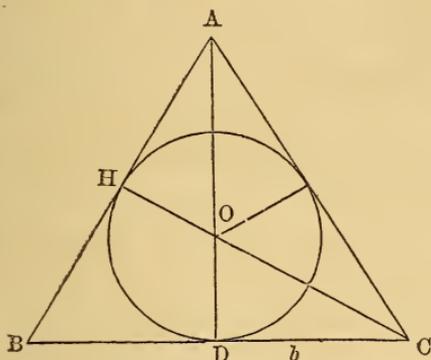
Whence,  $AP.PB = r^2 - x^2$

Or,  $x = \sqrt{r^2 - AP.PB}.$

That is the distance from the center to the required point will be found by *subtracting the rectangle from the square of the radius of the circle, and extracting the square root of the remainder.*

PROBLEM XXXI.

If each of the two equal sides of an isosceles triangle be represented by  $(a)$ , and the base by  $(2b)$ , what will be the radius of the inscribed circle in terms of  $a$  and  $b$ .



Let  $ABC$  be the triangle, and

$$AB=AC=a, \quad BD=b.$$

$$\text{Then } AD=\sqrt{a^2-b^2}.$$

$$\text{Place } OD=R=HO.$$

Now, by the similar triangle  $AHO$ ,  $ABD$ , we have

$$AO : OH :: AB : DB$$

That is,

$$AD-R : R :: a : b$$

And,

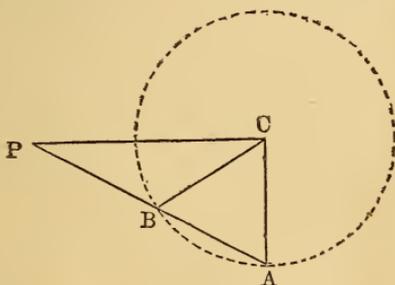
$$b\sqrt{a^2-b^2}-bR=aR$$

Whence,

$$R=\frac{b\sqrt{a^2-b^2}}{a+b}=\frac{b\sqrt{a-b}}{\sqrt{a+b}}.$$

PROBLEM XXXII.

From a point without a circle whose diameter is  $(d)$ , a line equal to  $(d)$  is drawn to the concave terminating in the concave arc, and bisected at the first point of meeting the circumference. Required the distance of the point without from the center of the circle.



Let  $C$  be the center of the circle, and  $P$  a point without, and  $AP=d$ , the diameter of the circle, which is bisected by the circumference at  $B$ .

Join  $CA$ ,  $CB$ , and  $CP$ .

Because  $AP$  is equal to the diameter of the circle, and it is

bisected in  $B$ ,  $AB$  must be equal to the radius of the circle; that is,  $AB=BC=CA$ , and the triangle  $ABC$  equiangular.

Also  $BP=BC$ , therefore the angle  $BCP=BPC$ . But the angle  $CBP=120^\circ$ , therefore the angle at  $P$  is  $30^\circ$ , and the angle  $PCA=90^\circ$ .

$$\text{Hence,} \quad \overline{CA}^2 + \overline{PC}^2 = \overline{PA}^2.$$

$$\text{Or,} \quad \left(\frac{1}{2}d\right)^2 + \overline{CP}^2 = d^2.$$

$$\text{Whence,} \quad \overline{CP}^2 = \frac{3}{4}d^2, \text{ or, } CP = \frac{d}{2}\sqrt{3}.$$

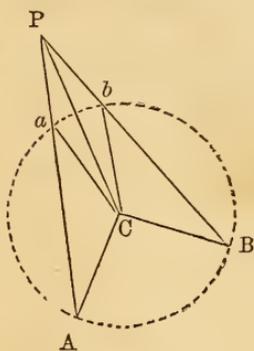
the value of the line sought.

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### MISCELLANEOUS PROPOSITIONS.

(Page 238.)

(3.) *If from any point without a circle, two straight lines be drawn to the concave part of the circumference, making equal angles with the lines joining the same point and the center; the parts of these lines which are intercepted within the circle, are equal.*



Let  $P$  be any point without a circle, and  $C$  the center of the circle. Join  $PC$  and draw  $PA, PB$  making equal angles with  $PC$ .

Join  $CA, CB, Ca$  and  $Cb$ . We are to prove that  $Aa=Bb$ .

In the two triangles  $PCA, PCB$ , we have  $CA=CB$ , and  $PC$  common to both triangles, and the angle  $CPA$  opposite the side  $AC$  is equal to  $CPB$  opposite the equal side  $BC$ .

That is, the two triangles have three parts respectively

equal. Therefore, the three other parts are also equal, and  $PA=PB$ . In like manner by the two triangles  $PCa, PCb$ , we prove  $Pa=Pb$ .

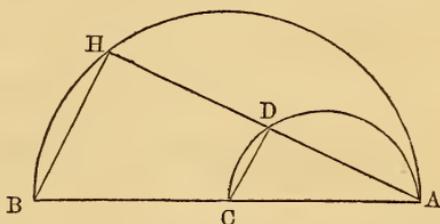
Whence, by subtraction,

$$PA - Pa = PB - Pb.$$

That is,  $Aa=Bb$ , and the theorem is demonstrated.

(4.) *If a circle be described on the radius of another circle, any straight line drawn from the point where they meet, to the outer circumference, is bisected by the interior one.*

Let  $AC$  be the radius of one circle and the diameter of another, as represented in the figure. From the point of contact  $A$ , of the two cir-



cles, draw any line, as  $AH$ ; this line is bisected in  $D$ . Join  $DC$  and  $HB$ . Then  $ADC$  being in a semicircle, is a right angle; also,  $AHB$  is a right angle, for the same reason: therefore,  $DC$  and  $HB$  are parallel. Whence,

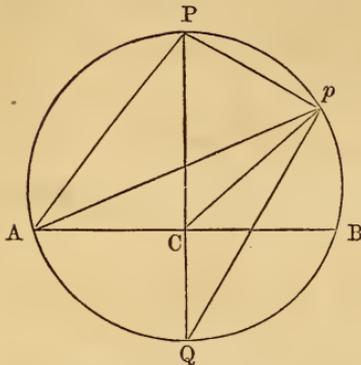
$$AD : AH :: AC : AB$$

But as  $AB$  is the double of  $AC$ , therefore  $AH$  is the double of  $AD$ , or  $AH$  is bisected in  $D$ , which was to be shown.

(5.) *From two given points on the same side of a line given in position, to draw two straight lines, which shall contain a given angle and be terminated in that line.*

Let  $AB$  be the line given in position, and  $P$  and  $p$  the given points. To make the problem definite, we take the given angle at  $35^\circ$ .

Join  $Pp$ , and draw  $PpQ$  a right angle. Take  $35^\circ$  from

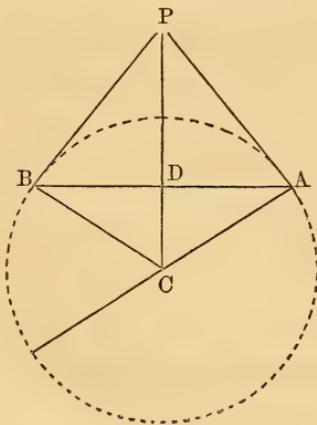


$90^\circ$ , and the remainder is  $55^\circ$ . Now, at  $P$ , make the angle  $pPQ$  equal to  $55^\circ$ , then the third angle of the triangle  $PQp$  will be  $35^\circ$ , the given angle. But it is not located in the line  $AB$ .

About the triangle  $PQp$ , describe the circle, and whence it cuts the lines  $AB$ , draw  $PA$ ,  $pA$ , and the angle  $PAp = PQp$ , because they are angles in the same segment of a circle; and the angle is located as was required. The point  $B$  is another in the line when the angle would be the same in magnitude.

SCHOLIUM.—The given angle must be less than the angle  $PCp$ , otherwise the two lines from  $P$ ,  $p$ , making the angle, would meet before they could reach the line given in position.

(6.) *If from any point without a circle, lines be drawn touching it, the angle contained by the tangents is double the angle contained by the line joining the points of contact, and the diameter drawn through one of them.*

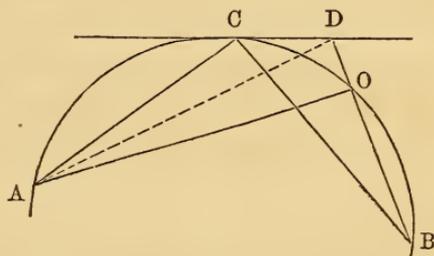


Let  $P$  be a point without a circle, and  $PA$ ,  $PB$ , tangents to the circle. Lines joining the center to the point of tangency, make right angles with the tangent lines. (Th. IV., Book III.). Join also  $PC$  and  $AB$ ,  $PC$  is common to the two right angled triangles  $PAC$  and  $PBC$ . Hence,  $PB = PA$ , and the angle at  $P$  is bisected by  $PC$ .

$AD$  is a line drawn from the right angle of the triangle  $PAC$  perpendicular to its hypotenuse  $PC$ , and divides that triangle into two similar triangles (Th. XXV., Book II., 1). Hence the angle  $DAC$  is equal to the angle  $APC$ , or  $BPA$ , is double of the angle  $BAC$ , which was to be demonstrated.

(7.) *If from any two points in the circumference of a circle, there be drawn two straight lines to a point in a tangent to that circle, they will make the greatest angle when drawn to the point of contact.*

Let  $A$  and  $B$  be the two points in the circle, and  $CD$  a tangent line. The proposition requires us to demonstrate that the angle  $ACB$  is greater



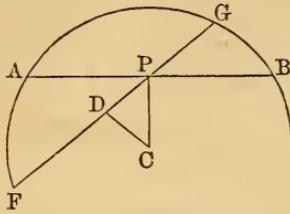
than the angle  $ADB$ .  $ACB = AOB$ , (Th. IX., Book III., Cor.) But  $AOB$  is greater than  $ADB$ , (Book I., Th. XII., Cor. 1). Therefore,  $ACB$  is also greater than  $ADB$ .

(8.) *From a given point within a given circle, to draw a straight line which shall make with the circumference an angle less than any angle made by any other line drawn from that point.*

NOTE.—An angle between a chord and a circumference is the same as between the chord and a tangent drawn through the same point. Thus the angle made by the chord  $AB$ , and the circumference at  $B$  is the same as between  $AB$  and a tangent drawn through  $B$ .

Let  $C$  be the center of the circle,  $P$  a given point within

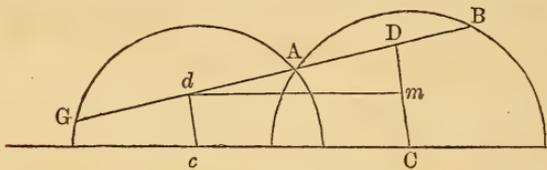
the circle not at the center. Join  $CP$ , and through  $P$  at right angles to  $CP$  draw the chord  $AB$ .



The angle made between  $AB$  and the circumference  $B$  is measured by half the arc  $BGA$ , and this is less than any other angle made by any other line drawn through  $P$ , and the circumference.

Through  $P$  draw any other line as  $FPG$ . Now the angle which  $PG$  makes with the circumference is measured by half the arc  $GAF$ . But  $GAF$  is a greater arc than  $BGA$ , because  $GF$  is greater than  $AB$ .  $GF$  we know to be greater than  $AB$ , because  $CD$  is less than  $CP$ ,  $CP$  being the hypotenuse of the right angled triangle  $CDP$ . Consequently the angle which  $PB$  makes with the circumference is less than that which  $PG$  makes with the circumference.

(9.) *If two circles cut each other, the greatest line that can be drawn through either point of intersection, is that which is parallel to the line joining their centers.*



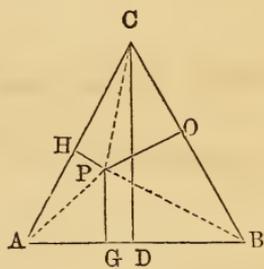
Let  $c$  and  $C$  be the centers of two circles which cut each other, and  $A$ , one of the points of intersection. Now our object is to prove that a line drawn through  $A$ , parallel to  $cC$ , will be greater than any other line, as  $GB$  drawn through  $A$ , and not parallel to  $cC$ .

From  $cC$  draw perpendiculars  $cd$ ,  $CD$ , to  $GB$ .  $AB$  is bisected in  $D$ , and  $AG$  in  $d$ , therefore,  $dD$ , is half  $GB$ . But  $dD$  is less than  $dm = cC$ , because  $dm$  is the hypotenuse of the right angled triangle  $dDm$ .

But if  $GB$  were drawn parallel to  $cC$  through  $A$ , then  $dD$  would be equal to  $cC$ . Consequently a line drawn through  $A$ , parallel to  $cC$ , is the greatest possible through  $A$ , which was to be shown.

(10.) *If from any point within an equilateral triangle perpendiculars be drawn to the sides, they are together equal to a perpendicular drawn from any of the angles to the opposite side.*

Let  $ABC$  be the equilateral  $\triangle$ ,  $CD$  a perpendicular from one of the angles on the opposite side; then the area of the  $\triangle$  is expressed by  $\frac{1}{2}AB \times CD$ . Let  $P$  be any point within the triangle, and from it let drop the three perpendiculars  $PG$ ,  $PH$ ,  $PO$ .



The area of the triangle  $APB$  is expressed by  $\frac{1}{2}AB \times PG$ . The area of the  $\triangle CPB$  is expressed by  $\frac{1}{2}CB \times PO$ : and the area of the  $\triangle CPA$  is expressed by  $\frac{1}{2}CA \times PH$ . By adding these three expressions together, (observing that  $CB$  and  $CA$  are each equal to  $AB$ ), we have for the area of the whole  $\triangle ACB$ ,  $\frac{1}{2}AB(PG+PH+PO)$ .

Therefore,  $\frac{1}{2}AB \times CD = \frac{1}{2}AB(PG+PH+PO)$

Dividing by  $\frac{1}{2}AB$ , gives  $CD = PG+PH+PO$ .

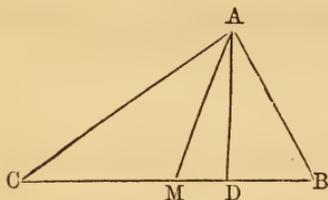
(11.) *If the points bisecting the sides of any triangle be joined, the triangle so formed will be one-fourth of the given triangle.*

If the points of bisection be joined, the triangle so formed will be similar to the given  $\triangle$ , (Th. XX., Book II.)

Then, the area of the given  $\triangle$  will be to the area of the  $\triangle$  formed by joining the bisecting points, as the square of a

line is to the square of its half; that is,  $2^2$  to 1, or as 4 to 1. Hence the  $\triangle$  formed is  $\frac{1}{4}$  of the given triangle.

(12.) *The difference of the angles at the base of any triangle, is double the angle contained by a line drawn from the vertex perpendicular to the base, and another bisecting the angle at the vertex.*



Let  $ABC$  be a  $\triangle$ . Draw  $AM$  bisecting the vertical angle, and draw  $AD$  perpendicular to the base.

*The theorem requires us to prove that the difference between the angles  $B$  and  $C$  is double of the angle  $MAD$ .*

By hypothesis, the angle  $CAM = MAB$ .

That is,  $CAM = MAD + DAB$ . (1)

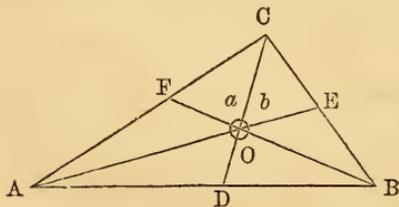
By (Th. XII., B. I., Cor. 4)  $\left\{ \begin{array}{l} C + CAM + MAD = 90^\circ \\ B + DAB = 90^\circ \end{array} \right\}$  (2)

Therefore,  $B + DAB = C + CAM + MAD$  (4)

Taking the value of  $CAM$  from (1), and placing it in (4), gives  $B + DAB = C + MAD + DAB + MAD$ .

Reducing  $(B - C) = 2MAD$ , which verifies the text.

(13.) *If from the three angles of a triangle, lines be drawn to the middle of the opposite sides, these lines will intersect each other in the same point.*



Let  $ABC$  be a  $\triangle$ , bisect  $BC$  in  $E$ ,  $AC$  in  $F$ .

Join  $AE$  and  $BF$ , and through their point of intersection  $O$ , draw the line  $CD$ . Now if we prove  $AD = DB$ , the theorem is true.

Triangles whose bases are in the same line, and vertices in the same point, are to one another as their bases ; and when the bases are equal, the triangles are equal. For this reason the  $\triangle AFO = \triangle FCO$ , and the  $\triangle COE = \triangle EOB$ .

Put  $\triangle AFO = a$  ; then  $\triangle FCO = a$ . Also, put  $\triangle COE = b$ , as represented in the figure.

Because  $CB$  is bisected in  $E$ , the  $\triangle ACE$  is half of the whole  $\triangle ABC$ . Because  $AC$  is bisected in  $F$ , the  $\triangle BFC$  is half the whole  $\triangle ABC$ .

That is,  $2a + b = 2b + a$ .

Whence,  $a = b$ , and the four triangles above the point  $O$  are equal to each other.

Let the area of the  $\triangle ADO$  be represented by  $x$ , and the area of  $DOB$  by  $y$ .

Now taking  $CD$  as the base of the triangles, we have

$$2a : x :: CO : OD$$

$$\text{Also, } 2b = 2a : y :: CO : OD$$

$$\text{Whence, } 2a : x :: 2a : y. \text{ Or, } x = y.$$

$$\text{Therefore, } AD = DB.$$

SCHOLIUM.—If the triangle,  $ABC$  be regarded as a thin lamina of matter, *its center of gravity* must be somewhere in the line  $AE$  ; for the part  $AEC =$  the part  $AEB$ . For a similar reason, the center of gravity must be somewhere in the line  $BF$ . Hence it must be at their intersection,  $O$ .

Again, the triangle  $AEB =$  the triangle  $ADC$ . And since  $x = y$ , and  $a = b$ , we have,

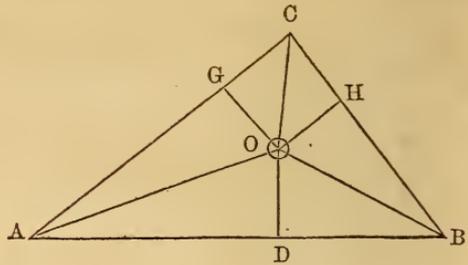
$$2x + a = 2a + x, \text{ or, } a = x.$$

$$\text{Hence, } \triangle AOB = \frac{1}{3} \triangle ACB ; \text{ and } OD = \frac{1}{3} CD.$$

(14.) *The three straight lines which bisect the three angles of a triangle, meet in the same point.*

Let  $ABC$  be any triangle, bisect two of its angles  $A$  and  $B$ , the bisecting lines meeting at  $O$ .

Let fall perpendiculars from  $O$  on the three sides,  $OD, OG, OH$ .

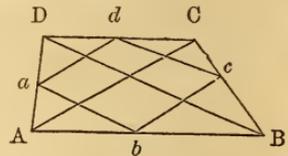


Join  $CO$ , and now if we can demonstrate that  $CO$  bisects the angle  $C$  the proposition will be proved. The two right angled triangles  $AOD, AOG$  have a common hypotenuse,  $AO$ , and equal angles respectively, therefore,  $OD=OG$ . In like manner, by the two right angled triangles,  $DOB$  and  $OHB$ , we prove  $OD=OH$ . Whence,  $GO=OH$ , and the hypotenuse  $CO$  is common to each of the triangles,  $CGO$  and  $CHO$ , therefore their angles are equal, and the angle  $C$  is bisected by the line  $CO$ . Hence the bisecting lines all meet at the point  $O$ , and the proposition is proved.

COR.—Let the student observe that  $OD, OG, OH$ , each, is equal to the radius of the inscribed circle.

(16.) *The figure formed by joining the points of bisection of the sides of a trapezium, is a parallelogram.*

Let  $ABCD$  be a trapezium. Draw the diagonals  $AC, BD$ . Bisect the sides in  $a, b, c$ , and  $d$ . Join  $abcd$ . We are to prove that this figure is a parallelogram.



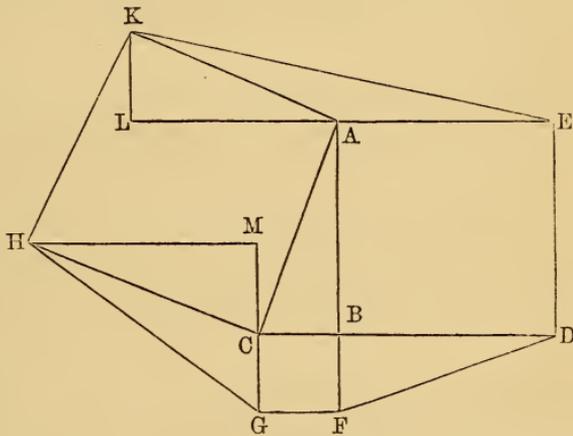
$ABD$  is a  $\triangle$  whose sides are bisected in  $a$  and  $b$ ; therefore,  $ab$  is parallel to  $BD$ , (Th. XVII., Book II.) In the same manner we can prove that  $dc$  is parallel to  $BD$ . Consequently  $ab$  and  $dc$  are parallel. It may likewise be shown

that  $ad$  and  $bc$  are parallel. Hence the figure  $abcd$  is a parallelogram.

(17.) *If squares be described on the three sides of a right angled triangle, and the extremities of the adjacent sides be joined, the triangles so formed are equivalent to the given triangle, and to each other.*

Let  $ABC$  be the original right angled triangle; on its sides describe the squares  $AD$ ,  $BG$ , and  $AH$ .

Join  $KE$ ,  $DF$ ,  $HG$ , thus forming the triangles  $KAE$ ,  $DBF$ , and  $HCG$ , and we are



to prove each of them equal to the triangle  $ABC$ , and to each other.

We will now prove the triangle  $ALK$  equal to  $ABC$ , as follows,

The angle  $LAB=90^\circ$ ,

Also  $KAC=90^\circ$ .

From each of these equals take away the common angle  $LAC$ , and the remaining angles  $KAL$  and  $CAB$  must be equal. Again,  $ALK$  is right angled at  $L$ , and  $ABC$  right angled at  $B$ , and the hypotenuse  $AK$  of the triangle  $ALK$ , is equal to  $AC$  the hypotenuse of  $ABC$ . Therefore,  $ALK$  is equal to  $ABC$ .

Whence,  $AL=AB=AE$ , and  $KL=BC$ .

Now, because  $AL=AE$ , the area of the triangle  $KAE$  is

equal to the area  $ALK$ , or  $ABC$ , and this is one of the facts which was to be demonstrated.

In like manner because  $ACH=90^\circ$ , and  $BCM=90^\circ$ , taking away the common angle  $MCA$  from each of these equals, we have the angle  $HCM=ACB$ ; but  $HC=AC$  and the triangles right angled, one at  $M$ , the other at  $B$ . Therefore, the triangle  $HCM$  is equal in all respects to  $ABC$ .

That is,  $CM=CB$ , but  $CB=CG$ , therefore  $MC=CG$ ; consequently the area of the triangle  $HCG=$  the area  $HCM$ , or  $ABC$ .

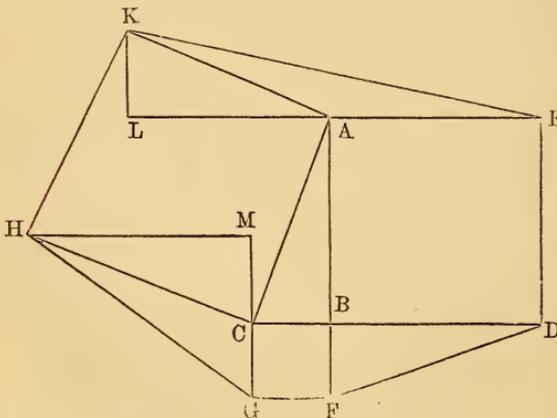
Now, because the area  $KAL=ABC$

And the area  $HCG=ABC$

Therefore, area  $KAL=$  the area  $HCG$ .

The triangle  $BDF=ABC$ , because the opposite angles at  $B$  are equal, each  $90^\circ$ .  $AB=BD$ , and  $CB=BF$ ; and the proposition is fully demonstrated.

(18.) *If squares be described on the hypotenuse and sides of a right angled triangle, and the extremities of the sides of the former, and the adjacent sides of the others, be joined, the sum of the squares of the lines joining them, will be five times the square of the hypotenuse.*



The two triangles  $KLE$ , and  $HMG$  are right angled triangles, right angles at  $L$ , and at  $M$ . Observe also that  $AE=AB=AL$ .

Whence,  
 $EL=2AB$ .

Also,  $KL=BC$ ,  $BC=CG=CM$ . Whence,  $MG=2BC$ .

But  $\overline{EL}^2 + \overline{KL}^2 = \overline{EK}^2$

And  $\overline{MG}^2 + \overline{MH}^2 = \overline{HG}^2$

That is,  $(2AB)^2 + (BC)^2 = (EK)^2$  (1)

And  $(2BC)^2 + (AB)^2 = (HG)^2$  (2)

By adding (1) and (2), observing that  $(2AB)^2$  is  $4\overline{AB}^2$

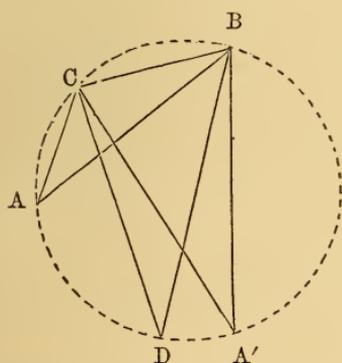
$$5\overline{AB}^2 + 5\overline{BC}^2 = (EK)^2 + (HG)^2.$$

That is, 5 times  $(AC)^2 = (EK)^2 + (HG)^2$

which verifies the proposition.

COR.—The area of the whole figure is double the square on  $AC$ , and 4 times the area of the triangle  $ABC$ .

(19.) *The vertical angle of an oblique angled triangle, inscribed in a circle, is greater or less than a right angle, by the angle contained between the base and the diameter drawn from the extremity of the base.*



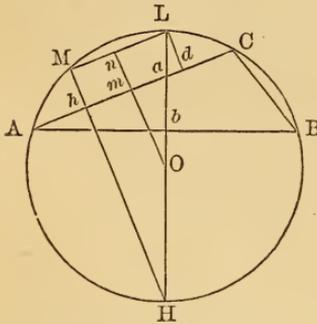
Let  $ACB$  be the vertical angle of a triangle, its base the chord  $AB$ . From the extremity of the base  $B$ , draw the diameter of the circle  $BD$ . Join  $CD$ .

Because  $BD$  is a diameter the angle  $BCD$  is a right angle; therefore, the angle  $ACB$  is greater than a right angle by the angle  $ACD$ . But  $ACD = ABD$ , each being measured by half the arc  $AD$ .

That is,  $ACB$  is greater than a right angle by the angle  $ABD$ , which was to be demonstrated.

Or, let  $A'BC$  be the triangle in a circle, and  $A'CB$  its vertical angle. This is less than a right angle, by the angle  $A'CD$ , or  $A'DD$ .

(20.) *If the base of any triangle be bisected by the diameter of its circumscribing circle, and from the extremity of that diameter, a perpendicular be let fall upon the longer side, it will divide that side into segments, one of which will be equal to half the sum, and the other to half the difference of its sides.*



Let  $ABC$  be the  $\triangle$ , bisect its base by the diameter of the circle drawn at right angles to  $AB$ .

From the center  $O$  let fall  $Om$  at right angles to  $AC$ , it will then bisect  $AC$ . From the extremity of the diameter  $H$ , draw  $Hh$  perpendicular to  $AC$ , and conse-

quently parallel to  $Om$ . Produce  $Hh$  to  $M$  and join  $ML$ . Complete and letter the figure as represented.

The two triangles  $Aab$  and  $Hha$  are equiangular. The angle  $a$  is common to them, and each has a right angle by construction, therefore the angle  $H =$  the angle  $A$ , and the arc  $ML =$  the arc  $CB$ , (Th. IX., Book III., C.); therefore  $CB = ML$ . The angle  $HML$  is a right angle, because it is in a semicircle, therefore,  $ML$  is parallel to  $AC$ , and  $ML$  is bisected in  $n$ .

$$\text{Now } Am = \frac{1}{2}AC. \quad nL = md = \frac{1}{2}ML = \frac{1}{2}BC.$$

$$\text{Therefore by addition, } Am + md = \frac{1}{2}(AC + CB).$$

$$\text{Or,} \quad Ad = \frac{1}{2}(AC + CB.)$$

But, if  $Ad$  is the half sum of the sides,  $dC$  or  $Ah$  must be the half difference; for the half sum and half difference make the greater of any two quantities.

(21.) *A straight line drawn from the vertex of an equilateral triangle, inscribed in a circle, to any point in the opposite circumference, is equal to the sum of the two lines drawn from the extremities of the base to the same point.*

Let  $ABC$  be an equilateral triangle inscribed in a circle, and designate each side by  $a$ , as represented in the figure.

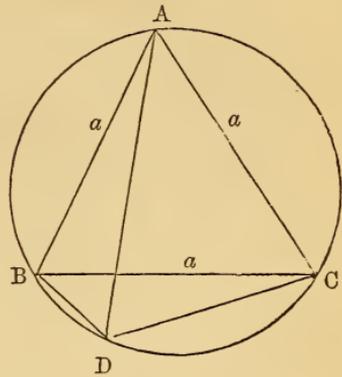
Take any point  $D$  on the arc  $BC$ , and join  $AD$  and designate it by  $x$ . Join  $BD$  and  $DC$ .

We are required to demonstrate that  $x = BD + DC$ .

Observe that  $ABDC$  is a quadrilateral inscribed in a circle, and by (Th. XXI., Book III.), we must have,

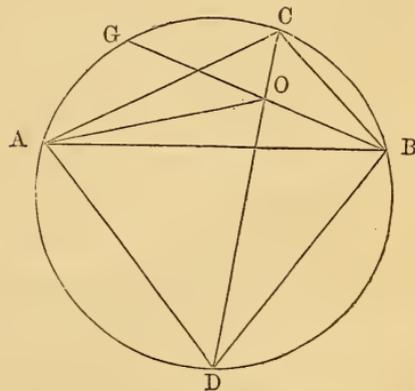
$$ax = a(BD) + a(DC).$$

Dividing each side by  $a$ , and  $x = BD + DC$ , which was to be demonstrated.



(22.) *The straight line bisecting any angle of a triangle inscribed in a circle, cuts the circumference in a point equidistant from the extremities of the side opposite to the bisected angle, and from the center of the circle inscribed in the triangle.*

Let  $ABC$  be a triangle in a circle, and draw  $CD$  bisecting the angle at  $C$ , and draw  $BG$  bisecting the angle at  $B$ . The intersection of these two bisecting lines at  $O$ , is the center of an imaginary circle inscribed in the triangle  $ABC$ .



The line  $CD$  bisects the angle at  $C$  by hypothesis, therefore, the arc  $AD =$  the arc

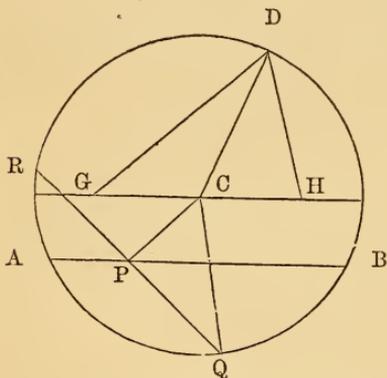
$BD$ . Hence the chord  $AD =$  the chord  $BD$ , and thus the first part of the theorem is proved.

In like manner, because  $BG$  bisects the angle  $B$ , the arc  $AG =$  the arc  $GC$ .

Now in the triangle  $DBO$ , the angle  $DBO$  or  $DBG$  is measured by half the sum of the arc,  $AD + AG$ . And the angle  $DOB$  is measured by half of the sum of the arcs  $DB + GC$ . But  $AD + AG = DB + GC$ . Therefore, the angles  $DBO$  and  $DOB$  are equal, and the triangle is isosceles, and  $DB = DO$ .

In like manner we can prove that  $DA = DO$ , and the second point of the theorem is demonstrated.

(23.) *If from the center of a circle, a line be drawn to any point in a chord of an arc, the square of that line, together with the rectangle of the segments of the chord, will be equal to the square on the radius of the circle.*



Let  $C$  be the center of a circle, and  $P$  any point in the chord  $AB$ . Join  $PC$ , then we are required to prove that

$$\overline{PC}^2 + AP \cdot PB = \overline{CQ}^2.$$

Through  $P$  draw the chord  $RQ$  at right angles to  $PC$ , and join  $CQ$ .

Because  $CP$  is a line perpendicular from the center to a chord it bisects the chord. Hence,  $RP = PQ$ .

And because two chords intersect at  $P$ , we have,

$$AP \cdot PB = RP \cdot PQ = \overline{PQ}^2.$$

Add  $\overline{CP}^2$  to each member of this last equation, and we have,

$$\overline{CP}^2 + AP.PB = \overline{PQ}^2 + \overline{PC}^2$$

But by the right angled triangle  $PCQ$ , we perceive that the second member is equal to  $\overline{CQ}^2$ . Therefore,

$$\overline{CP}^2 + AP.PB = \overline{CQ}^2,$$

which was to be demonstrated.

(24.) *If two points be taken in the diameter of a circle equidistant from the center, the sum of the squares of the two lines drawn from these points to any point in the circumference, will always be the same.*

Take the same circle as in the preceding theorem, and let  $GC = CH$ . Take any point in the circumference as  $D$ , and join  $DG$ ,  $DC$ ,  $DH$ .

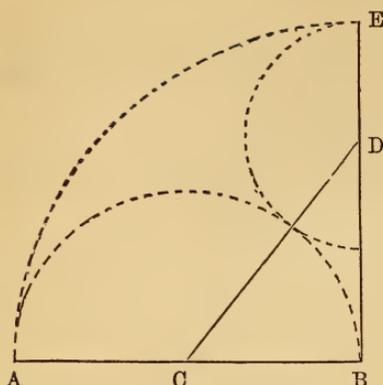
Now because the base of the triangle  $GDH$  is bisected in  $C$ , we have by (Th. XLII., Book I.),

$$\overline{GD}^2 + \overline{DH}^2 = 2\overline{DC}^2 + 2\overline{GC}^2.$$

But at whatever point on the circumference  $D$  may be placed,  $DC$  and  $GC$  will always retain the same value. Therefore, the sum of the squares of the other two sides is constantly equal to the same sum.

(25.) *If on the diameter of a semicircle, two equal circles be described, and in the space enclosed by the three circumferences, a circle be inscribed, its diameter will be two thirds the diameter of either of the equal circles.*

Let  $AB$  be the diameter of one of the two equal circles and construct the figure as here represented.



We are required to show the relation between  $CB$  and  $ED$ .

Place  $CB=r$  and  $ED=x$ .

Then  $BE=2r$ , and  $BD=2r-x$ ,  $CD=r+x$ .

In the right angled triangle  $CBD$ , we have,

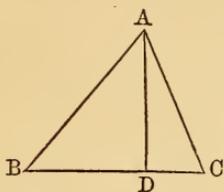
$$\overline{CB}^2 + \overline{BD}^2 = \overline{CD}^2.$$

That is,  $r^2 + (2r-x)^2 = (r+x)^2$ ,

Or,  $r^2 + 4r^2 - 4rx + x^2 = r^2 + 2rx + x^2$

Whence,  $6rx = 4r^2$ , and  $x = \frac{2}{3}r$ .

(26.) *If a perpendicular be drawn from the vertical angle of any triangle to the base, the difference of the squares of the sides is equal to the difference of the squares of the segments of the base.*



Let  $ABC$  be a triangle. From  $A$  let fall  $AD$  perpendicular on  $BC$ , making two right angled triangles  $ADB$ ,  $ADC$ . These triangles give

$$(AD)^2 + (DB)^2 = (AB)^2$$

$$(AD)^2 + (DC)^2 = (AC)^2$$

By subtraction  $(\overline{BD})^2 - (DC)^2 = (AB)^2 - (AC)^2$ .

This equation demonstrates the theorem.

COR.—By factoring the above equation, we have,

$$(BD+DC)(BD-DC) = (AB+AC)(AB-AC).$$

Observing that  $BD+DC=BC$ , and changing this equation into a proportion, we have,

$$BC : AB+AC = AB-AC : (BD-DC).$$

(This is Prop. 6, Plane Trigonometry, Page 256, Geometry.)

This proportion is true whatever relation exist between  $AB$  and  $AC$ . When  $AB=AC$ , then  $BD=DC$  and the preceding proportion becomes

$$BC : 2AB : : 0 : 0.$$

This is an apparent absurdity, but we can reconcile it to truth by taking the product of the extremes and means, and we have,

$$(BC)0=(2AB)0.$$

An equation obviously true,

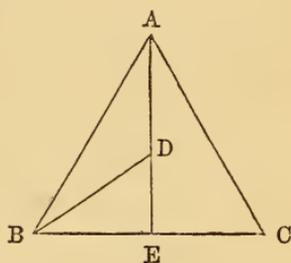
Or,  $\frac{BC}{2AB}=\frac{0}{0}$ , and 0 divided by 0 is any quantity whatever,

Hence,  $0 : 0 : : a : \text{to any quantity whatever.}$

(27.) *The square described on a side of an equilateral triangle is equal to three times the square of the radius of the circumscribing circle.*

Let  $ABC$  be an equilateral triangle. Let fall the perpendicular  $AE$  on the base, that line will bisect the angle  $A$ . Draw  $BD$  bisecting the angle at  $B$ .

We will now prove  $AD=BD$ ; then  $D$  must be the center of the circumscribing circle.



Each angle of an equilateral triangle is  $60^\circ$  ( $\frac{1}{3}$  of  $180^\circ$ ). Bisecting each of these angles, we have,  $DBA$ ,  $DAB$ , each  $30^\circ$ , and therefore  $AD=DB$ , and hence, if  $D$  be taken as the center of circle with  $DB$  or  $DA$  as radius, that circle will circumscribe the triangle. Put  $AB=2a$ , then  $BE=a$ .

Also, place  $BD=x$ , then  $DE=\frac{1}{2}x$ . (It being the side of a right angled triangle opposite  $30^\circ$ ). (Prob. 2 of this Key.)

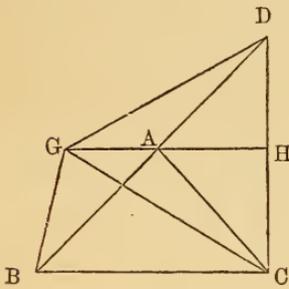
Now by the right angled triangle  $DBE$ , we have,

$$(BE)^2 + (DE)^2 = (BD)^2.$$

That is,  $a^2 + \frac{1}{4}x^2 = x^2$ , or,  $3x^2 = 4a^2$ .

But  $(4a^2) = (AB)^2$ . That is,  $3x^2 = (AB)^2$ , which demonstrates the theorem.

(28.) *The sum of the sides of an isosceles triangle, is less than the sum of the sides of any other triangle on the same base, and between the same parallels.*



Let  $ABC$  be the isosceles triangle.  $AB=AC$ . Through the point  $A$  draw  $GAH$  parallel to  $BC$ .

Take  $G$  any other point on the line  $GH$ , and draw  $BG$  and  $GC$ .

We are to show that  $AB+AC$  is less than  $BG+GC$ . Produce  $AB$  to  $D$ , making  $AD=AB$ , or  $AC$ .

Then by reason of the parallels  $GH$  and  $BC$ , the angle  $DAH$  is equal to the angle  $ABC$ , and  $HAC=ABC$ .

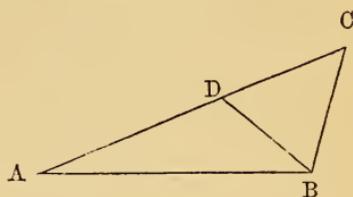
Because  $AD=AC$ , the angle  $ADH=$  the angle  $ACH$ .

Whence the two triangles  $ADH$  and  $ACH$ , are equal in all respects, and  $GH$  is perpendicular to  $DC$ ; whence any point in the line  $GH$  is equally distant from the two points  $D$  and  $C$ .

Now the straight line  $BD=BA+AC$ , and because  $DG=GC$ ,  $DG+GB=GB+GC$ . But  $DG+GB$ , the two sides of a  $\triangle$  are greater than the third side  $DB$ ; therefore  $GB+GC$  is greater than  $BD$ , that is, greater than  $BA+AC$ .

(29.) *In any triangle, given one angle, a side adjacent to the given angle, and the difference of the other two sides, to construct the triangle.*

Draw  $AB$  equal to the given side. From one extremity of the base  $A$ , draw  $AC$  indefinitely, making the angle  $BAC$ , equal to the given angle. (Prob. V., Book IV.)

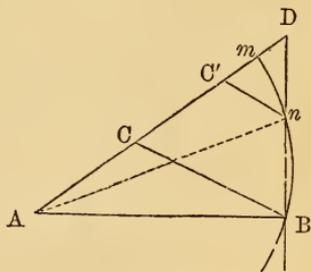


Take  $AD$  equal to the given difference between the other two sides and join  $DB$ . From the point  $B$  make the angle  $DBC$  equal to the angle  $BDC$ , and the point  $C$  will be the vertex of the triangle required. Because  $BC=CD$ , and  $AD$  was made equal to the given difference of two sides, and  $AB$  the given side.

(30.) *In any triangle, given the base, the sum of the other two sides, and the angle opposite the base, to construct the triangle.*

Take any point  $A$ , and from it draw the line  $AD$ , equal to the sum of the sides.

At the extremity  $D$ , make the angle  $ADB$  equal to half of the given angle opposite the base. From  $A$  as a center, with the distance  $Am$  equal to the given base, describe an arc,  $mnB$ ,



cutting  $DB$  in  $n$  and  $B$ .

From  $B$  draw  $BC$  making the angle  $DBC$ , equal to the

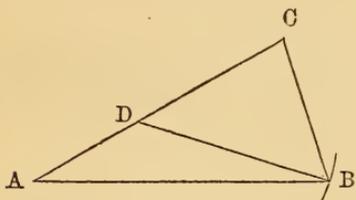
angle  $D$ . Also from  $n$  draw  $nC'$  parallel to  $BC$ , then  $ABC$  and  $AnC'$ , are triangles corresponding to the given conditions.

For,  $AB=An=Am$ , the given base. The angles  $ACB$ , or  $AC'n$  opposite the respective bases,  $AB, An$ , are equal to the given angle, because  $ACB$  being the exterior angle of the triangle  $CDB$ , is equal to the sum of  $D$  and  $DBC$ ; but  $DBC$  was made equal to  $D$  by construction, and  $D$  was made equal to half the given angle.

The same reasoning applies to the triangle  $DC'n$ .

Also,  $CD=CB$  by construction, therefore  $AC+CB=AC+CD$ , or  $AD$  the sum of the given sides.

(31.) *In any triangle, given the base, the angle opposite to the base, and the difference of the other two sides, to construct the triangle.*



Let  $ABC$  be the triangle. To discover its proper or direct construction, we must suppose the problem solved, and then analyze it.

If  $AB$  is the given base, and  $AD$  the given difference of the other sides, it is then obvious that  $DC=CB$ , and knowing the angle  $C$  which is given, we can determine each of the other angles of the triangle  $CBD$ .

Therefore,  $CDB$  is known, and its supplement  $ADB$  is known. Whence the angle

$$ADB=90^{\circ}+\frac{1}{2}C.$$

Take any point  $A$  and draw  $AC$  indefinitely. From the

scale of equal parts take  $AD =$  to the given difference, and make the angle

$$ADB = 90^\circ + \frac{1}{2}C.$$

Take the given base from the scale of equal parts in the dividers, and with one foot on  $A$  as a center, strike an arc, cutting  $DB$  in  $B$ . The line  $AB$  will be the base.

From the point  $B$ , and from the line  $BD$  make the angle  $DBC = BDC$ , and produce  $DC$  and  $BC$  until they meet in  $C$ , and we have the triangle  $ABC$ , as was required.

# PLANE TRIGONOMETRY.

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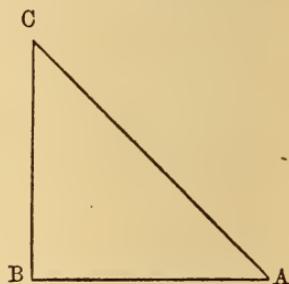
## SECTION II.

(PAGE 289.)

NOTE.—In each of the following examples in this section, there are six parts, three angles  $A$ ,  $B$ , and  $C$ , and the three corresponding sides  $a$ ,  $b$ ,  $c$ ,  $A$  being opposite to  $a$ , etc. Each of the following examples may be referred to one and the same triangle. The right angle always at  $B$ , and in right angled trigonometry this part is always given, and not generally expressed. The right angle and two other parts being given, the remaining three parts can be determined.

Ex. 2. Given  $AC$ , 73.26, and the angle  $A$ ,  $49^\circ 12' 20''$ , to find the other parts.

From  $90^\circ$  take  $A$ , and we have  $C=40^\circ 47' 40''$ .



As $\sin. 90^\circ$ .	10.000000		10.000000
To $\sin. 49^\circ 12' 20''$	9.879129	cos.	9.815144
So is 73.26	1.864867	log.	1.864867
To $BC$ , 55.462	1.743996		1.680011
	<i>Ans.</i> $BC$ , 55.46.		$AB$ , 47.87.

Ex. 3. Given  $AB$ , 469.34, and the angle  $A$ ,  $51^\circ 26' 17''$ , to find the other parts.

From  $90^\circ$  take  $A$ , and  $C=38^\circ 33' 43''$ .

	<i>For BC.</i>		<i>For AC.</i>
As sin. $C$ , $38^\circ 33' 43''$	9.794739	sin. $C$	9.794739
Sin. $A$	9.893171	sin. $90^\circ$	10.000000
$AB$ , 469.34	2.671488	$AB$	2.671488
(Sin. $A$ ) ( $AB$ )	<u>12.564659</u>	$AC$	<u>2.876749</u>
$BC$ , 588.74	2.769920	$AC$	752.92. <i>Ans.</i>

Ex. 4. Given  $AB$ , 493, and the angle  $C$ ,  $20^\circ 14'$ , to find the other parts.

The remaining angle  $A$  is of course,  $69^\circ 46'$ .

	<i>For CB.</i>		<i>For AC.</i>
As sin. $C$ , $20^\circ 14'$	9.538880	sin. $C$	9.538880
Cos. $C$ , or sin. $A$	9.972338	sin. $90^\circ$	10.000000
$AB$ , 493	2.692847	$AB$	2.692847
	<u>12.665185</u>	$AC$ , $1425.5=$	<u>3.153967</u>
$CB$ , 1337.53	3.126305		

Ex. 5. Given  $AB$ , 331, and the angle  $A$ ,  $49^\circ 14'$ , to find the other parts.

The angle  $C=40^\circ 46'$ .

	<i>For CB.</i>		<i>For AC.</i>
As sin. $C$	9.814900	sin. $C$	9.814900
Cos. $C$ or Sin. $A$	9.879311	sin. $90^\circ$	10.000000
$AB$	2.519828	$AB$	2.519828
	<u>12.399139</u>	$AC$ , 506.91	<u>2.704928</u>
$CB$ , 383.92	2.584239		

Ex. 6. Given  $AC$ , 45, and the angle  $C$ ,  $37^\circ 22'$ , to find the other parts.

The angle  $A$  must be  $52^\circ 38'$ .

	<i>For CB.</i>		<i>For AB.</i>
As sin. $90^\circ$	10.000000	sin. $90^\circ$	10.000000
Sin. $C$ , $37^\circ 22'$	9.783127	sin. $A$	9.900240
$AC$	1.653212	$AC$	1.653212
$AB$ , 27.311	<u>1.436339</u>	$CB$ , 35.764	<u>1.553452</u>

Ex. 7. Given  $AC$ , 4264.3, and the angle  $A$ ,  $56^\circ 29' 13''$ , to find the other parts.

The angle  $C$  must be  $33^\circ 30' 47''$ .

	<i>For CB.</i>		<i>For AB.</i>
As sin $90^\circ$	10.000000	sin. $90^\circ$	10.000000
Sin. $A$ , $56^\circ 29' 13''$	9.921041	sin. $C$	9.742038
$AC$	3.629848	$AC$	3.629848
$CB$ , 3555.4	<u>3.550889</u>	$AB$ , 2354.4	<u>3.371886</u>

Ex. 8. Given  $AB$ , 42.2, and the angle  $A$ ,  $31^\circ 12' 49''$ , to find the other parts.

The angle  $C$  must be  $58^\circ 47' 11''$ .

	<i>For BC.</i>		<i>For AC.</i>
As sin. $C$ , $58^\circ 47' 11''$	9.932088	sin. $C$	9.932088
Sin. $A$ , $31^\circ 12' 49''$	9.714522	sin. $90^\circ$	10.000000
$AB$	1.625312	$AB$	1.625312
	<u>11.339834</u>	$AC$ , 49.34	<u>1.693223</u>
$BC$ , 25.57	<u>1.407746</u>		

Ex. 9. Given  $AB$ , 8372.1, and  $BC$ , 694.73, to find the other parts.

(By Prop. III. Plane Trig., 2d part, we have,)

$$8372.1 : 694.73 :: R : \tan. A.$$



Ex. 12. Given  $AC$ , 4824, and  $BC$ , 2412, to find the other parts.

In this example the hypotenuse is double of the side  $BC$ , therefore the angle opposite  $BC$  or  $A$ , must be  $30^\circ$ , and the angle  $C$ ,  $60^\circ$ .

If we call  $BC$ , 1,  $AC$ , 2, and  $AB$ ,  $x$ , then  $x^2 + 1 = 4$ , and  $x = \sqrt{3}$ . Whence,  $AB = 2412\sqrt{3} = 4178$ , nearly.

Ex. 13. In this example the hypotenuse of the right angled triangle is 94,770,000 miles, and the most acute angle only  $16' 6''$ , the double of the side opposite this small angle is required.

The difficulty here is to obtain the sine of  $16' 6''$  to a sufficient degree of accuracy from the common table. This is explained on page 288, Text-book. Or we may operate as follows,

As	Sin. $90^\circ$		10.000000
Is to	94,770,000	log.	7.976671
So is	Sin. $16' 6'' = (16.1) \sin. 1'$		6.463726, sin. $1'$
	Log. 16.1		1.206826
	443850		5.647223
	2		
Sun's diameter	887700		

Ex. 14. Solution, As	94,779,000		7.976671
	$R$		10.000000
	Log. 3956		3.597256
Sine of solar parallax =			5.620585 = $8.61''$ . Ans.

Ex. 15. The distance from the earth to the moon, at mean distance in miles, is  $(60.3)(3960)$ .

Statement, As  $\sin. 90^\circ : \sin. 15' 32'' :: (60.3)(3960) : M.$

Observe that  $15' 32'' = 932''$ ,  $\sin. 932'' = 932$  times  $\sin. 1''$ .

*Solution.*

Log. 60.3		1.780317
Log. 3960		3.597695
Sin. 1'' (see page 288, Geom.)		4.685575
Log. 932		2.969416
Sum, less 10,	<i>R</i> , 1078.9	3.033003
	$\frac{2}{2157.8}$	
Diameter of moon,	2157.8	<i>Ans.</i>

## OBLIQUE ANGLED TRIGONOMETRY.

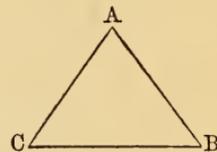
(PRACTICAL PROBLEMS.)

(Page 295.)

NOTE.—One triangle will serve for the solution of nearly all the following examples in this section. But the learner should draw his figure as near as he can according to the data given in each example. He will thus be less likely to make mistakes than he otherwise would.

Let  $ABC$  be any oblique angled triangle.

Ex. 1. Given  $AB$ , 697, the angle  $A$ ,  $81^\circ 30' 10''$ , and the angle  $B$ ,  $40^\circ 30' 44''$ , to find the other parts.



Because the sum of the three angles make two right angles, or  $180^\circ$ . Therefore,

$$C = 180^\circ - (81^\circ 30' 10'' + 40^\circ 30' 44'') = 57^\circ 59' 6''.$$

As  $\text{Sin. } C : AB :: \text{sin. } A : BC.$

Whence,	To sine $A$ , $81^\circ 30' 11''$	9.995207
	Add, 697	2.843233
		12.838440
	Sub. sin. $C$ , $57^\circ 59' 6''$	9.928350
	$BC$ , 813	log. 2.910090
Again,	Sin. $C : AB :: \sin. B : AC$	
	Log. $AB$	2.843233
	Sin. $B$	9.812652
		12.655885
	Sin. $C$ , as before	9.928350
	$AC$ , 534	2.727535

Ex. 2. Given  $AC$ , 720.8, the angle  $A$ ,  $70^\circ 5' 22''$ , and  $B$ ,  $59^\circ 35' 36''$ , to find the other parts.

The angle  $C$  must be  $50^\circ 19' 2''$ .

$$\text{As} \quad \sin. B : AC :: \sin. A : BC = \frac{AC \cdot \sin. A}{\sin. B}.$$

$$\text{Also,} \quad \sin. B : AC :: \sin. C : AB = \frac{AC \cdot \sin. C}{\sin. B}.$$

Sin. $A$	9.973232	sin. $C$	9.886260
$AC$ , 720.8	2.857815		2.857815
	12.831047		12.744075
Sin. $B$	9.935737	sin. $B$	9.935737
$BC$ , 785.8	2.895310	$AB$ , 643.2	2.808338

Ex. 3. Given  $BC$ , 980.1, the angle  $A$ ,  $7^\circ 6' 26''$ , and the angle  $B$ ,  $106^\circ 2' 23''$ , to find the other parts.

The angle  $C$  must be  $66^\circ 51' 11''$ .

$$\text{As} \quad \sin. A : BC :: \sin. B : AC = \frac{BC \cdot \sin. B}{\sin. A}.$$

$$\sin. A : BC :: \sin. C : AB = \frac{BC \cdot \sin. C}{\sin. A}.$$

Sin. $B$	9.982755	sin. $C$	9.963552
$BC$	2.991270	$BC$	2.991270
	<u>12.974025</u>		<u>12.954822</u>
Sin. $A$	9.092463	sin. $A$	9.092463
$AC, 7613.1$	3.881562	$AB, 7283.8$	3.862359

Ex. 4. Given  $AB, 896.2$ ,  $BC, 328.4$ , and the angle  $C, 113^\circ 45' 20''$  to find the other parts.

$$\text{As } AB : \sin. C :: BC : \sin. A = \frac{BC. \sin. C}{AB}.$$

Sin. $C = \cos. 23^\circ 45' 20''$	9.761550
$BC, 328.4$	2.516403
	<u>12.477953</u>
$AB, 896.2$	2.952405
Sin. $A = \sin. 19^\circ 35' 46''$	9.525548

Now having the angles  $A$  and  $C$ , we subtract their sum from  $180^\circ$ , and thus find the remainder  $B, 46^\circ 38' 54''$ . *Ans.*

With this angle we determine  $AC$ .

Ex. 5. Given  $AC=4627$ ,  $BC, 5169$ , and the angle  $A, 70^\circ 25' 12''$ , to find the other parts.

$$\text{As } BC : \sin. A :: AC : \sin. B = \frac{AC. \sin. A}{BC}.$$

Sin. $A, 70^\circ 25' 12''$	9.974132
$AC, 4627$	3.665299
	<u>13.639431</u>
$BC, 5169$	3.713407
Sin. $B, 57^\circ 29' 56''$	9.926024

We now have the angles  $A$  and  $B$ , subtracting their sum from  $180^\circ$ , gives  $C=52^\circ 4' 52''$ . *Ans.*

For  $AB$ , we have,

$$\text{Sin. } A : BC :: \text{sin. } C : AB = \frac{BC \cdot \text{sin. } C}{\text{sin. } A}.$$

$BC$	3.713407.
$\text{Sin. } C, 52^\circ 4' 52''$	9.897011
	13.610418
$\text{Sin. } A$	9.974132
$AB, 4328$	3.636286

EX. 6. Given  $AB$ , 793.8,  $BC$ , 481.6,  $AC$ , 500, to find the angles.

NOTE.—Here all three sides are given, and the solution is by formulas found on page 259, Geometry.

The angles of a triangle are denoted by  $A, B, C$ , and the sides opposite by  $a, b, c$ .  $A$  opposite  $a$ , etc.

The formulas are  $\frac{1}{2}(a+b+c)=S$ .

$$\text{Cos.}^2 \frac{1}{2}A = \frac{S(S-a)}{bc} ; \text{cos.}^2 \frac{1}{2}B = \frac{S(S-b)}{ac} ; \text{cos.}^2 \frac{1}{2}C = \frac{S(S-c)}{ab} ;$$

the radius being unity.

By logarithms the formulas become

	Log. $(\text{cos.}^2 \frac{1}{2}A) = \log. S(S-a) - \log. bc.$	&c.	
{	$a = 481.6$	Log. $S$	2.948266
{	$b = 500$	Log. $(S-a)$	2.608633
{	$c = 793.8$	Log. $S(S-a)$	5.556899
	2)1775.4	log. $b, 2.698970$	
	887.7 = $S$	log. $c, 2.899711$	
	$a, 481.6$	5.598681	5.598681
	$406.1 = (S-a)$	log. $(\text{cos.}^2 \frac{1}{2}A)$	-1.958218 (2, div.)
		log. $(\text{cos.}^2 \frac{1}{2}A)$	-1.979109
			10.
			9.979109
		2	
	$A =$	$35^\circ 15' 32''.$	

Having thus found one of the angles, the other may be determined by the direct proportion between the sides and the sines of the opposite angles.

Ex. 7. Given  $AB$ , 100.3,  $AC$ , 100.3, and  $BC$ , 100.3, to find the angles.

Here the sides are equal to each other, the triangle is therefore equiangular as well as equilateral. Consequently each angle is one third of  $180^\circ$ , or  $60^\circ$ .

Ex. 8. Given  $AB$ , 92.6,  $BC$ , 46.3, and  $AC$ , 71.2, to find the angles.

NOTE.—In example 6 we obtained the angle  $A$ , for radius unity. Here we will find the angle  $B$ , for radius  $R$ , whose log. is 10. The formula must be

$$\text{Cos. } \frac{1}{2}B = \left( \frac{R^2 S(S-b)}{ac} \right)^{\frac{1}{2}}.$$

By logarithms this becomes,

$$\text{Log. } (\text{cos. } \frac{1}{2}B) = \frac{1}{2}(\text{log. } R^2 S(S-b) - \text{log. } ac).$$

$b = 71.2$		$\text{Log. } R^2 = 20$		
{	$a = 46.3$		$\text{log. } S$	2.021396
	$c = 92.6$		$\text{log. } (S-b)$	1.529559
	$\underline{2)210.1}$	$\text{log. } a,$	1.665581	$\underline{23.550955}$
	$105.05 = S$	$\text{log. } c,$	1.966611	
	$71.2$		$\underline{3.632192}$	$3.632192$
	$\underline{33.85 = S-b.}$			$\underline{2)19.918763}$
		$\text{Cos. } \frac{1}{2}B = \text{cos. } 24^\circ 23' 45''$		$\underline{9.959381}$
			2	
		$B =$	$\underline{48^\circ 47' 30''.}$	

$$\begin{array}{r} \text{Now,} \quad b : \sin. B :: a : \sin. A = \frac{a \sin. B}{b}. \\ \text{Log. } a = \quad \quad \quad 1.665581 \\ \text{Sin. } B = \quad \quad \quad 9.876402 \\ \hline \quad \quad \quad \quad \quad \quad 11.541983 \\ \text{Log. } b = \quad \quad \quad 1.852480 \\ \text{Sin. } A = \sin. 29^\circ 17' 22'' \quad \quad \quad 9.689503 \\ \hline \text{Hence,} \quad \quad C = 101^\circ 55' 8'' \end{array}$$

Ex. 9. Given  $AB$ , 4963,  $BC$ , 5124,  $AC$ , 5621, to find the angles.

$a = 5124$	For the angle $C$	
$b = 5621$	$\log. R^2$	20.
$c = 4963$	$\log. S$	3.895091
<u>15708</u>	$\log. (S-c)$	<u>3.461048</u>
$S = 7854$	$\log. a$ , 3.709609	27.356139
$S-c = 2891$	$\log. b$ , 3.749814	
	<u>7.459423</u>	<u>7.459423</u>
		2)19.896716
	$\cos. \frac{1}{2}C = \cos. 27^\circ 23' 27.5''$	<u>9.948358</u>
	$C = 54^\circ 46' 55''$	

For the angle  $A$

$$\begin{array}{r} c : \sin. C :: a : \sin. A. \\ \text{Sin. } C \quad \quad \quad 9.912202 \\ \text{Log. } a \quad \quad \quad 3.709609 \\ \hline \quad \quad \quad \quad \quad \quad 13.621811 \\ \text{Log. } c \quad \quad \quad 3.695744 \\ \text{Sin. } A = \sin. 57^\circ 30' 28'' \quad \quad \quad 9.926067 \end{array}$$

Ex. 10. Given  $AB$ , 728.1,  $BC$ , 614.7,  $AC$ , 583.8 to find the angles.

$a=614.7$		$\log. R^2$	20.000000
$b=583.8$	}	$\log. 2.766264$	2.983762
$c=728.1$		$\log. 2.862191$	2.542327
1926.6		5.628455	25.526089
$S$ 963.3			5.628455
614.7			2)19.897634
$(S-a)$ 348.6		$\cos. \frac{1}{2}A = \cos. 27^\circ 16' 26''$	9.948817
		2	
	$A$	54° 32' 52''	

Now as  $a : \sin. A :: b : \sin. B = \frac{b \sin. A}{a}$ .

$\log. b$		2.766264	
$\sin. A$		9.910944	
		12.677208	
$\log. a, 614.7$		2.788663	
$\sin. B = \sin. 50^\circ 40' 58''$		9.888545	

Ex. 11. Given  $AB, 96.74, BC, 83.29,$  and  $AC, 111.42,$  to find the angles.

$a= 83.29$		$\log. R^2$	20.000000
$b=111.42$	}	$\log. S$	2.163534
$c= 96.74$		$\log. (S-a)$	1.795428
2)291.45		$\log. b, 2.046963$	23.958962
$S$ 145.725		$\log. c, 1.985606$	
$a$ 83.29		4.032569	4.032569
$(S-a)$ 62.435			2)19.926393
		$\text{Cos. } \frac{1}{2}A = \cos. 23^\circ 15' 22.5'' =$	9.963196
		2	
	$A =$	46° 30' 45''	

As	$a : \sin. A :: b : \sin. B$	
	Sin. $A =$	9.860652
	$b$	2.046963
		11.907615
	$a$	1.920593
	Sin. $B$ , sin. $76^\circ 3' 46''$	9.987022

$A$  and  $B$  being as above,  $C$  must be  $57^\circ 25' 29''$ .

Ex. 12. Given  $AB$ , 363.4,  $BC$ , 148.4, and the angle  $B$ ,  $102^\circ 18' 27''$ , to find the other parts.

Here we have two sides and their included angle given, and we apply (Prop. VII., Plane Trig., page 257), which is

$$\left. \begin{array}{l} 363.4 \\ 148.4 \end{array} \right\} \text{sum.} : \left. \begin{array}{l} 363.4 \\ 148.4 \end{array} \right\} \text{diff.} :: \tan. \left( \frac{C+A}{2} \right) : \tan. \left( \frac{C-A}{2} \right) \quad (1)$$

We observe that  $C$  must be greater than the angle  $A$ , because  $C$  is opposite the greater side.

$$511.8 : 215 :: \tan. 38^\circ 50' 47'' : \tan. \frac{C-A}{2}.$$

Log. 215	2.332438
----------	----------

Tan. $\frac{C+A}{2}$ , $38^\circ 50' 47''$	9.905986
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Log. 511.8	2.709100
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Tan. $\frac{1}{2}(C-A)$ $18^\circ 41' 29'' =$	9.529324
---	----------

$\frac{1}{2}(C+A)$ $38^\circ 50' 47''$	
--	--

Sum $C$ $57^\circ 32' 16''$	} <i>Ans.</i>
Diff. $A$ $20^\circ 9' 18''$	

As Sin.  $C : c(363.4) :: \sin. B : b.$

Sin. $B$ , $102^\circ 18' 27'' = \cos. 12^\circ 18' 27''$	9.989903
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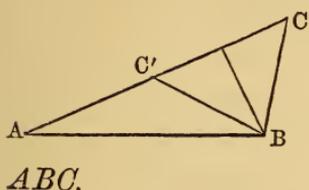
log. 363.4	2.560385
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sin. $C$ , $57^\circ 32' 16''$	9.926211
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log. $AC = \log. 420.8$	2.624077
-------------------------	----------

Ex. 13. Given  $AB$ , 632,  $BC$ , 494, and the angle  $A$ ,  $20^\circ 16'$ , to find the other parts, the angle  $C$  being acute.



If no mention were made requiring the angle  $C$  to be acute, the data would give the triangle  $ABC'$ , for the solution as well as the triangle  $ABC$ .

In such cases it is customary to solve both triangles, and call the solution ambiguous.

$$\text{As,} \quad BC : AB :: \sin. A : \sin. C$$

$$\text{Or,} \quad 494 : 632 :: \sin. 20^\circ 16' : \sin. C$$

Log. $\sin. 20^\circ 16'$	9.539565
log. 632	<u>2.800717</u>
	12.340282
log. 494	<u>2.693727</u>
$C, 26^\circ 18' 19''$	9.646555

$A$  and  $C$  taken from  $180^\circ$ , gives  $133^\circ 25' 41''$ , for  $B$ . Lastly for  $AC$ , we have,

$$\sin. A : BC :: \sin. B : AC (b).$$

$\sin. B, 133^\circ 25' 41'' = \cos. 43^\circ 25' 41''$	9.861079
log. $BC$	<u>2.693727</u>
	12.554806
$\sin. A$	<u>9.539565</u>
$AC, 1035.7$	3.015241

Ex. 14. Given  $AB$ , 53.9,  $AC$ , 46.21, and the angle  $B$ ,  $58^\circ 16'$  to find the other parts.

$$\text{As,} \quad 46.21 : \sin. B, 58^\circ 16' :: 53.9 : \sin. C$$

$\sin. B, 58^\circ 16'$	9.929677
$AB, 53.9$	<u>1.731589</u>
	11.661266
$AC, 46.21$	<u>1.664736</u>
$\sin. C, 82^\circ 46'$	9.996530

$B$  and  $C$  being known, their sum,  $141^\circ 2'$  taken from  $180^\circ$ , gives  $38^\circ 58'$ , for the angle  $A$ . *Ans.*

As,	sin. $C$ : $AB$ , 53.9 : sin. $A$ , $38^\circ 58'$ : $BC$	
	Sin. $38^\circ 58'$	9.798560
	$AB$ , 53.9	1.731589
		11.530149
	Sin. $C$ , $82^\circ 46'$	9.996530
	$BC$ , 34.16	1.533619

**Ex. 15.** Given  $AB$ , 2163,  $BC$ , 1672, and the angle  $C$ ,  $112^\circ 18' 22''$ , to find the other parts.

As, sin.  $C$  :  $AB$ , 2163 : : sin.  $A$  :  $BC$ , 1672.

Whence, Sin.  $A = \frac{1672 \sin. C}{2163}$ .

Sin. $C = \cos. 22^\circ 18' 22''$		9.966221
	log. 1672	3.223236
		13.189457
	log. 2163	3.335057
	Sin. $A$ , $45^\circ 39' 22''$	9.854400

Now,  $A$  and  $C$  being known,  $B$  must be  $22^\circ 2' 16''$ .

Sin. $A$ : 1672 : : sin. $B$ , $22^\circ 2' 16''$ : $AC$ .		
	Sin. $22^\circ 2' 16''$	9.574283
	$BC$ , 1672	3.223236
		12.797519
	Sin. $A$ , $45^\circ 39' 22''$	9.854400
	$AC$ , 877.2	2.943119

**Ex. 16.** Given  $AB$ , 496,  $BC$ , 496, and the angle  $B$ ,  $38^\circ 16'$  to find the other parts.

In this example we observe that  $AB = BC$ , therefore, the

angles  $A$  and  $C$ , must be equal to each other, and the value of each must be

$$\frac{1}{2}(180^\circ - 38^\circ 16') = 70^\circ 52'.$$

Now,  $\text{Sin. } 70^\circ 52' : 496 :: \text{sin. } 38^\circ 16' : AC.$

$$AC = \frac{496 \cdot \text{sin. } 38^\circ 16'}{\text{sin. } 70^\circ 52'}.$$

Sin. $38^\circ 16'$	9.791917
Log. 496	2.695482
	12.487399
Sin. $70^\circ 52'$	9.975321
$AC = 325.1$	2.512078

Ex. 17. Given  $AB$ , 428, the angle  $C$ ,  $49^\circ 16'$ , and  $(AC + CB) = 918$ , to find the other parts.

NOTE.—This problem is the same as 30, in Book VIII., Geometry, page 242, and its general solution is given in this Key, on page 59.

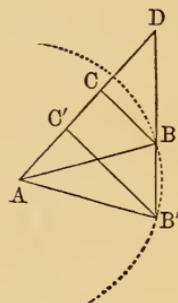
Either triangle,  $ABC$  or  $AB'C'$  will correspond with the data. We will take the triangle  $ABC$ , and commence by solving the triangle  $ABD$ .

Because  $CB = CD$ , the angle  $ACB$  is double of the angle  $D$ .

Therefore,  $D = 24^\circ 38'$ ,  $AD = 918$ ,  $AB = 428$ .

As  $428 : \text{sin. } D :: 918 : \text{sin. } ABD.$

Log. 918	2.962843
Sin. $24^\circ 38'$	9.619938
	12.582781
Log. 428	2.631444
Sin. $ABB'$ , $63^\circ 22' 48''$	9.951337



Or,	$ABD =$	116° 37' 12"
Sub.	$CBD =$	24° 38' 00"
Diff. =	$ABC =$	91° 59' 12"

To  $ABC$ , add  $ACB$ , 49° 16', and their sum, 141° 15' 12", taken from 180° gives  $A = 38° 44' 48''$ .

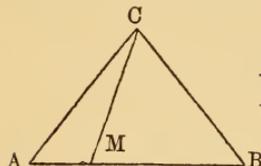
Lastly,  $\text{Sin. } C : AB :: \text{sin. } A : BC$ .

Which is,  $\text{Sin. } 49° 16' : 428 :: \text{sin. } 38° 44' 48'' : BC$ .

$\text{Sin. } 38° 44' 48''$	9.796490
$\text{Log. } 428$	2.631444
	12.427934
$\text{Sin. } 49° 16'$	9.879529
$BC, 353.5$	2.548405

Now, as  $(AC + BC) = 918$ , we have  $AC = 918 - 353.5 = 564.5$ . In like manner, solving the triangle  $AB'C'$ , we shall have  $AC' = 353.5$ , and  $B'C' = 564.5$ .

Ex. 18. Given a side and its opposite angle, and the difference of the other two sides, to construct the triangle and find the other parts.



Let  $ABC$  be the triangle.  $AC = 126$ ,  $B = 29° 46'$ , and  $AM$ , the difference between  $AB$  and  $BC$ , = 43.

From 180° take 29° 46' and divide the remainder by 2. This gives the angle  $BMC$  or  $BCM$ .  $BMC$  taken from 180°, gives  $AMC$ .

Now in the triangle  $AMC$ , we have the two sides  $AC$ , 126,  $AM$ , 43, and the angle  $AMC$ , to find the angle  $A$ . The computation is as follows:  $180° - 29° 46' = 150° 14'$ ; half, = 75° 7' =  $BMC$ .  $180° - 75° 7' = 104° 53' = AMC$ . Now in the  $\triangle AMC$ , we have

$$AC : AM :: \text{sin. } 104° 53' : \text{sin. } ACM$$

$$126 : 43 :: \cos. 14^\circ 53' : \sin. ACM$$

Cos. $14^\circ 53'$	9.985180
Log. 43	1.633468
	11.618648
Log. 126	2.100371
Sin. $ACM = \sin. 19^\circ 15' 28''$	9.518277

Now to  $BCM$ ,  $75^\circ 7'$  add  $ACM$ ,  $19^\circ 15' 28''$ , and we have  $ACB$ ,  $94^\circ 22' 28''$ . Subtracting  $19^\circ 15' 28''$  from  $75^\circ 7'$ , that is,  $ACM$  from  $BMC$ , and the difference must be

$$A = 55^\circ 51' 32''.$$

Lastly, As  $\sin. B : AC, 126 :: \sin. A, 55^\circ 51' 32'' : CB$ .

Sin. $A$	9.917851
126	2.100371
	12.018222
Sin. $B, 29^\circ 46'$	9.695892
$BC, 210.05$	2.322330

Ex. 19. Given  $AB, 1269, AC, 1837$ , and the including angle  $A, 53^\circ 16' 20''$ , to find the other parts.

Solution by the same formulas as in 12.

$b, 1837$	} sum :	$1837$	} diff. :: $\tan. \frac{1}{2}(B+C) : \tan. \frac{1}{2}(B-C)$ .
$c, 1269$		$1269$	
$\overline{3106}$		$\overline{568}$	$:: \tan. 63^\circ 21' 50'' : \tan \frac{1}{2}(B-C)$ .
		Tan. $63^\circ 21' 50''$	10.299685
		Log. 568	2.754348
			13.054033
		Log. 3106	3.492201
		$\frac{1}{2}(B-C), 20^\circ 1' 57''$	9.561832
But		$\frac{1}{2}(B+C) = 63^\circ 21' 50''$	
Sum =		$B = 83^\circ 23' 47''$ .	Diff. $C = 43^\circ 19' 53''$ .

Lastly,	Sin. $B : b :: \sin. A : a(BC)$	
	Sin. $A, 53^\circ 16' 20''$	9.903896
	Log. $b, 1837$	3.264109
		<u>13.168005</u>
	Sin. $B, 83^\circ 23' 47''$	9.997109
	$AC, 1482.16$	<u>3.170896</u>

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### SECTION III.

#### APPLICATION OF PLANE TRIGONOMETRY.

(Page 305.)

(1.) Required the height of a wall whose angle of elevation, at the distance of 463 feet is observed to be  $16^\circ 21'$ .

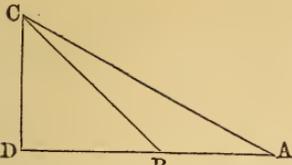
This example presents a right angled triangle, the base of which is 463, and the acute angle at the extremity of the base  $16^\circ 21'$ .

Let  $x =$  the required height ; then

$$R : \tan. 16^\circ 21' :: 463 : x = \frac{463 \tan. 16^\circ 21'}{R}$$

	Log. 463	2.665581
	Tan. $16^\circ 21'$	9.467413
<i>Ans.</i>	135.8	<u>2.132994</u>

(2.) We solve this example by the adjoining geometrical figure.



$ABC$  is a triangle,  $CBD$  its exterior angle.

Whence,  $CBD = ACB + A$ .

That is,  $31^\circ 18' = ACB + 26^\circ 18'$ ,  
 or,  $ACB = 5^\circ$ ,  $AB = 214$ .

$$\text{Sin. } 5^\circ : 214 :: \text{sin. } 26^\circ 18' : CB.$$

Whence,  $BC = \frac{214 \cdot \text{sin. } 26^\circ 18'}{\text{sin. } 5^\circ}$

Again, in the triangle  $CBD$  we have

$$\text{Sin. } 90^\circ : CB :: \text{sin. } 31^\circ 18' : CD.$$

That is,  $\text{Sin. } 90^\circ : \frac{214 \cdot \text{sin. } 26^\circ 18'}{\text{sin. } 5^\circ} :: \text{sin. } 31^\circ 18' : CD$ .

Whence,  $CD = \frac{214 \cdot \text{sin. } 26^\circ 18' \cdot \text{sin. } 31^\circ 18'}{\text{sin. } 90^\circ \cdot \text{sin. } 5^\circ}$ .

214	2.330414
Sin. $26^\circ 18'$	9.646474
Sin. $31^\circ 18'$	9.715602
Log. Num.	<u>21.692490</u>
Sin. $90^\circ$ 10.000000	
Sin. $5^\circ$ 8.940296	18.940296
$CD = 565.2$	<u>2.752194</u>

(3.) Here the perpendicular of a right angled triangle is given 149.5 feet, and the vertical angle  $57^\circ 21'$ . The opposite angle is therefore  $32^\circ 39'$ .

As,  $\text{Sin. } 32^\circ 39' : 149.5 :: \text{sin. } 57^\circ 21' : \text{Ans.}$

Or, As  $\text{Cos. } 57^\circ 21' : 149.5 :: \text{sin. } 57^\circ 21' : \text{Ans.}$

$$\text{Ans.} = 149.5 \cdot \frac{\text{sin. } 57^\circ 21'}{\text{cos. } 57^\circ 21'} = 149.5 \cdot \text{tan. } 57^\circ 21'.$$

149.5	2.174641
tan. $57^\circ 21'$	10.193307
$\text{Ans.} = 233.3$	<u>2.367948</u>

(4.) Here are two right angled triangles to be solved. The angle of depression, is equal to the angle opposite to the perpendicular.

Hence, for the distance of the mean object, we have the following proportion.

$$\begin{array}{r} \text{Sin. } 48^\circ 10' : 138 :: \text{cos. } 48^\circ 10' : \text{Dis.} \\ \text{Dis.} = 138. \frac{\text{cos. } 48^\circ 10'}{\text{sin. } 48^\circ 10'} = 138. \text{cot. } 48^\circ 10'. \\ \begin{array}{r} 138 \qquad \qquad \qquad 2.139879 \\ \text{cot. } 48^\circ 10' \qquad \qquad \qquad 9.951896 \\ \hline \text{Least dis. } 123.52 \qquad \qquad \qquad 2.091775 \\ \text{The greatest dis.} = 138. \text{cot. } 18^\circ 52' \end{array} \end{array}$$

(5.) Here is but one acute angled plane triangle, and the angle opposite to the base is found thus.

$$\begin{array}{r} 31^\circ 15' \\ 86^\circ 27' \\ \hline 117 \ 42 \\ 180 \\ \hline 62^\circ 18' \end{array}$$

Let  $x$  represent the distance from one extremity of the base to the house, and  $y$  represent the distance of the other extremity. Then

$$\begin{array}{r} \text{Sin. } 62^\circ 18' : 312 :: \text{sin. } 31^\circ 15' : x \\ \text{Sin. } 62^\circ 18' : 312 :: \text{sin. } 86^\circ 27' : y. \\ x = 312. \frac{\text{sin. } 31^\circ 15'}{\text{sin. } 62^\circ 18'}. \qquad y = 312. \frac{\text{sin. } 86^\circ 27'}{\text{sin. } 62^\circ 18'}. \\ \begin{array}{r} 312 \qquad 2.494155 \qquad 312 \qquad 2.494155 \\ \text{Sin. } 31^\circ 15' \qquad 9.714978 \qquad \text{sin. } 86^\circ 27' \qquad 9.999166 \\ \hline \qquad \qquad 12.209133 \qquad \qquad \qquad \qquad \qquad 12.493321 \\ \text{Sin. } 62^\circ 18' \qquad 9.947136 \qquad \text{sin. } 62^\circ 18' \qquad 9.947136 \\ x = 182.8 \qquad 2.261997 \qquad y = 351.7 \qquad 2.546185 \end{array} \end{array}$$

(6.) This is in many respects the same as (5), except in numbers.

Sin. $60^\circ$ : 260 :: sin. $40^\circ$ : $x$ .	
Log. 260	2.414973
Sin. $40^\circ$	9.808067
	12.223040
Sin. $60^\circ$	9.937531
192.8	2.285509

192.8 is the distance from one extremity of the base to the tree, but this line (192.8) makes an angle with the base of  $80^\circ$ . Let 192.8 be a hypotenuse of a right angled triangle, and the angles are  $80^\circ$  and  $10^\circ$ .

The side opposite  $80^\circ$  is the line or distance required.

Sin. $90^\circ$ : 192.8 :: sin. $80^\circ$ : perpendicular.	
192.8	2.285509
Sin. $80^\circ$	9.993351
<i>Ans.</i> 190.1	2.278860

(7.) Let  $BC$  be the eminence, 268 feet, and  $AD$  the steeple. Draw  $CE$  parallel to the horizontal  $AB$ . Then  $ECD=40^\circ 3'$ ,  $ECA=CAB=56^\circ 18'$ .

$DCA=56^\circ 18' - 40^\circ 3' = 16^\circ 15'$ ,  $DAC=90^\circ - 56^\circ 18' = 33^\circ 42'$ .

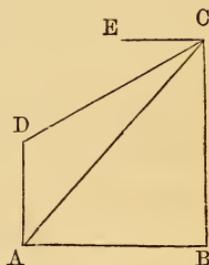
In the  $\triangle ABC$ , we have

$$\text{Sin. } 56^\circ 18' : 268 :: \text{sin. } 90^\circ : AC.$$

$$AC = \frac{268 \times R}{\text{sin. } 56^\circ 18'}$$

In the  $\triangle ADC$ , we have the supplement to the angle  $ADC$ , equal to  $16^\circ 15'$  added to  $33^\circ 42'$ , or  $49^\circ 57'$ ; therefore,

As  $\text{Sin. } ADC : AC :: \text{sin. } DCA : AD.$

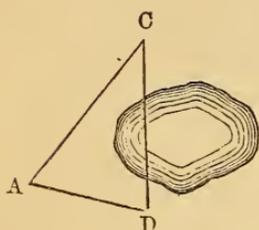


That is,  $\text{Sin. } 49^\circ 57' : \frac{268 \times R}{\text{sin. } 56^\circ 18'} :: \text{sin. } 16^\circ 15' : AD.$

Log.  $AD =$

$$\begin{aligned} \log. [268.R.\text{sin. } 16^\circ 15'] - \log. [\text{sin. } 49^\circ 57' . \text{sin. } 56^\circ 18'] = \\ [2.428135 + 10 + 9.446893] - [9.883936 + 9.920099] = \\ 21.875028 - 19.804035 = 2.070993 \end{aligned}$$

Whence,  $AD = 117.76$  feet.



(8.) Let  $C$  and  $D$  be the two objects, and  $A$  the point at which both can be seen.

$AD = 1428$ ,  $AC = 1840$ , and the angle at  $A$   $36^\circ 18' 24''$

From  $180^\circ$

Angles  $C$  and  $D = 143^\circ 41' 36''$

$\frac{1}{2}$  sum  $71^\circ 50' 48''$

Here we will apply the following theorem in trigonometry.

As the sum of two sides is to their difference, so is the tangent of half the sum of the angles at the base, to the tangent of half their difference.

Let  $x =$  the half difference between  $D$  and  $C$ .

Then,  $3268 : 412$

Or,  $817 : 103 :: \tan. 71^\circ 50' 48'' : \tan. x.$

Log. 103 2.012837

Tan.  $71^\circ 50' 48''$  10.484284

12.497121

817 2.912222

Tan.  $x = \tan. 21^\circ 1' 55''$  9.584899

$71^\circ 50' 48''$

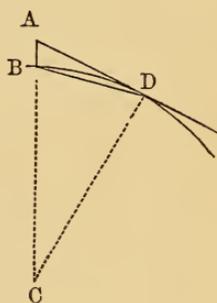
Angle  $D = 92^\circ 52' 43''$

Angle  $C = 50^\circ 48' 53''$

$$\text{Sin. } 50^\circ 48' 53'' : 1428 :: \text{sin. } 36^\circ 18' 24'' : CD.$$

	1428	3.154728
	Sin. $36^\circ 18' 24''$	9.772400
		12.927128
	Sin. $50^\circ 48' 53''$	9.889362
<i>Ans.</i>	1090.85	2.037766

(9.) Let  $AB$  represent the mountain, and  $AD$  the visible distance.  $AB$  produced will pass through the center of the earth at  $C$ . From  $D$  draw  $CD$  perpendicular to  $AD$ . Join  $BD$ .  $ADC$  is a right angled triangle.



$$CAD = 90^\circ - 2^\circ 13' 27'' = 87^\circ 46' 33''.$$

$$ACD = 2^\circ 13' 27''. \quad ADB = \frac{1}{2}ACD = 1^\circ 6' 43.5''.$$

$$ABD = 91^\circ 6' 43.5''.$$

Now in the  $\triangle ABD$ , we have

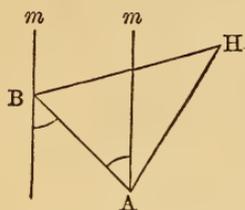
	Sin. $1^\circ 6' 43.5''$ : 3 :: sin. $91^\circ 6' 43.5''$ : $AD$ .	
	Sin. $91^\circ 6' 43.5'' = \cos. 1^\circ 6' 43.5''$	9.999919
	Log. 3	0.477121
		10.477040
	Sin. $1^\circ 6' 43.5''$	8.287976
	Log. $AD$ , 154.54	2.189064

In the triangle  $ADC$ , we have

	sin. $ACD$ : $AD$ :: cos. $ACD$ : $CD$ .	
	Cos. $ACD = \cos. 2^\circ 13' 27''$	9.999674
	$AD$	2.189064
		12.188738
	Sin. $ACD = \sin. 2^\circ 13' 27''$	8.588932
	$CD$	3.599806
	Log. 2	0.301030
	Diameter, log. 7958 miles,	3.900836

(10.) Let  $H$  be the location of the headland,  $A$  the position of the ship when the first observation was taken, and  $B$  its position when the last observation was taken.

$AB=20$  miles, and  $mA, mB$ , meridians passing through the eye of the observer at the two stations.



The data gives us the angle  $mAB = 47^\circ 49'$ ,  $mBH = 87^\circ 11'$ , the sum of these two angles taken from  $180^\circ$ , gives the angle  $ABH = 45^\circ$ .

The angle  $BAH$ , is obviously equal to the sum of  $47^\circ 49'$ , and  $39^\circ 23'$ , which is  $87^\circ 12'$ . The sum of  $87^\circ 12'$  and  $45^\circ$  taken from  $180^\circ$  gives  $AHB = 47^\circ 48'$ .

We now have  $AB=20$  miles one side, and each of the angles,  $A$ ,  $B$ , and  $H$ , to find  $AH$ , and  $BH$ .

$$\text{For } AH \quad \text{Sin. } 47^\circ 48' : 20 :: \text{sin. } 45^\circ : AH$$

$$\text{For } BH, \quad \text{sin. } 47^\circ 48' : 20 :: \text{sin. } 87^\circ 12' : BH.$$

Sin. $45^\circ$	9.849485	sin. $87^\circ 12'$	9.999481
Log. 20	1.301030		1.301030
	<u>11.150515</u>		<u>11.300511</u>
Sin. $47^\circ 48'$	9.869704		9.869704
$AH, 19.09$	<u>1.280811</u>	$BH, 26.96$	<u>1.430807</u>

(11.) By (Th. XVIII., Book III.), the length of a line drawn from the top of the tower to touch the surface of the sea, must be,  $\sqrt{7960.5280.100}$ , and from the same point on the sea, the line extended to the mast head of the ship must be  $\sqrt{7960.5280.90}$ . The problem requires the sum of these two lines.

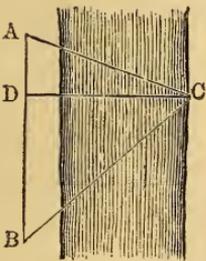
The computation by logarithms is as follows,

Log. 7960	3.900913	3.900913
Log. 5280	3.722634	3.722634
Log. 100	2.000000	log. 90 1.954243
	2)9.623547	2)9.577790
64829.6	4.811773	61502.8 4.788895
Sum in feet, 126332.4		log. 5.101514
Log. 5280		3.722634
Sum in miles,	23.92	1.378880
Add $\frac{1}{13}$ for refraction	1.84	
	25.76	<i>Ans.</i>

(12.) As  $\sin. 35^\circ : 143 \text{ feet} :: \cos. 35^\circ : \text{Dist.}$

Whence,  $\text{Dist.} = 143. \frac{\cos. 35^\circ}{\sin. 35^\circ} = 143. \cot. 35^\circ.$

Log. 143	2.155336
Cot. $35^\circ$	10.154773
<i>Ans.</i> 204.22	2.310109



(13.) Let  $CD$  be the breadth of the river, or the distance sought,  $AB=500$  yards, the measured base.

The angle  $ABC=53^\circ$ , and  $BAC, 79^\circ 12'$ . Whence the angle  $ACB=47^\circ 48'$ .

Now,

$$\sin. 47^\circ 48' : 500 :: \sin. 79^\circ 12' : BC.$$

Whence,  $BC = \frac{500 \cdot \sin. 79^\circ 12'}{\sin. 47^\circ 48'}$ .

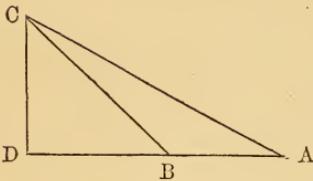
Again in the right angled triangle  $DBC$ , we have

$$\sin. D : BC :: \sin. B : DC.$$

That is,  $\text{Sin. } 90^\circ : BC :: \text{sin. } 53^\circ : DC.$

Whence,  $DC = \frac{500 \cdot \text{sin. } 79^\circ 12' \cdot \text{sin. } 53^\circ}{\text{sin. } 47^\circ 48' \cdot \text{sin. } 90^\circ}.$

Log. 500	2.698970
Sin. $79^\circ 12'$	9.992239
Sin. $53^\circ$	9.902349
	22.593558 numerator,
Log. $R$ , sin. $47^\circ 48'$	19.869704 denominator.
Log. $CD = \log. 529.48$	2.723854



(14.) Let  $L$  represent the length of the inclined plane, and  $P$  its perpendicular height.

The angle  $CBD = 46^\circ$ , and  $CAB = 31^\circ$ . Whence,  $ACB = 15^\circ$ , and  $AB = 200$ . The triangle  $ABC$  gives the proportion

$$\text{Sin. } 15^\circ : 200 :: \text{sin. } 31^\circ : L = \frac{200 \cdot \text{sin. } 31^\circ}{\text{sin. } 15^\circ}$$

From the triangle  $CBD$ , we obtain,

$$\text{Sin. } 90^\circ : L :: \text{sin. } 46^\circ : P = \frac{L \cdot \text{sin. } 46^\circ}{\text{sin. } 90^\circ}.$$

That is,  $P$  or  $CD = \left( \frac{200 \cdot \text{sin. } 31^\circ}{\text{sin. } 15^\circ} \right) \frac{\text{sin. } 46^\circ}{\text{sin. } 90^\circ}.$

Log. 200	2.301030
Log. sin. $31^\circ$	9.711839
Log. sin. $46^\circ$	9.856934
Log. of numerator	21.869803
Log. of denominator	19.412996 = $R \cdot \text{sin. } 15^\circ$
Log. 286.28	2.456807

(15.) Take the same triangle as in the preceding example.

$$\text{Then, } P = \frac{300 \cdot \sin. 32^\circ \cdot \sin. 58^\circ}{R \cdot \sin. 26^\circ}, \text{ Dis.} = \frac{300 \cdot \sin. 32^\circ \cdot \cos. 58^\circ}{R \cdot \sin. 26^\circ}$$

Log. 300	2.477121		2.477121
Sin. 32°	9.724210		9.724210
Sin. 58°	9.928420	cos. 58°	9.724210
Log. num.	22.129751		21.925541
Log. denom.	19.641842		19.641842
$P=307.54$	2.487909	dis. 192.18	2.283699

(16.) Here we have a triangle, one side of which is 440 yards, and the adjacent angles,  $83^\circ 45'$ , and  $85^\circ 15'$ , therefore the angle opposite must be  $11^\circ$ .

Now, for the side of the triangle which is opposite the angle  $85^\circ 15'$ , we have the following proportion.

$$\sin. 11^\circ : 440 :: \sin. 85^\circ 15' : x.$$

For the other side, we have

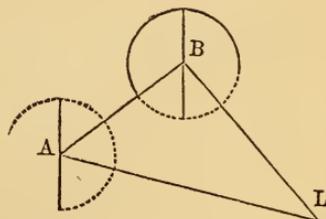
$$\sin. 11^\circ : 440 :: \sin. 83^\circ 45' : y.$$

	<i>For x.</i>		<i>For y.</i>
Log. 440	2.643453		2.643453
Sin. $85^\circ 15'$	9.998506	sin. $83^\circ 45'$	9.997411
	12.641959		12.640864
Sin. $11^\circ$	9.280599		9.280599
2298.05	3.361360	2292.26	3.360265

(17.) Let  $A$  and  $B$  be the positions of the ship, when the observations were taken.

Then  $AB=12$  miles in the direction north east.

Then, also  $AL$ , is the direction to the land, and  $BL$ , is another direction to the same point.



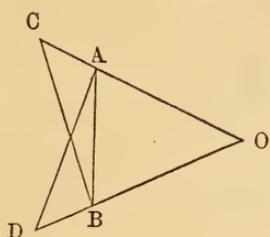
The angle  $LAB$  is 5 points of the compass, or  $56^\circ 15'$ , and the angle  $ABL$ , is 9 points, or  $101^\circ 15'$ .

Hence, the angle at  $L$  must be  $22^\circ 30'$ .

Now,  $\text{Sin. } 22^\circ 30' : 12 :: \text{sin. } 56^\circ 15' : BL$ .

$$BL = \frac{12 \cdot \text{sin. } 56^\circ 15'}{\text{sin. } 22^\circ 30'}$$

Log. 12	1.079181
Sin. $56^\circ 15'$	9.919846
	10.999027
Sin. $22^\circ 30'$	9.582840
<i>Ans.</i> 26.072	1.416187



(18.) This problem requires the adjoining figure.

We must compute the angle  $CAB$ , to find  $BAO$ .

And from the triangle  $ABD$ , we must compute the angle  $ABD$ , to find  $ABO$ .

We compute the angle  $CAB$ , by the following formula (Prop. 8, Plane Trigonometry.)

$$\text{Cos. } \frac{1}{2}A = \sqrt{\frac{R^2 s(s-a)}{bc}}$$

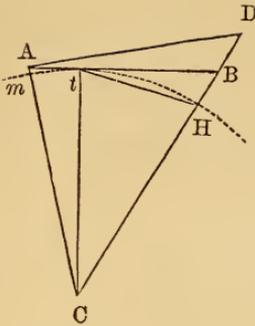
For the triangle  $ABC$ ,

$$a=560, b=100, c=500.$$

Whence,  $s=580$ , and  $s-a=20$ .

$$\text{Cos. } \frac{1}{2}A = \sqrt{\frac{R^2 580 \times 20}{100 \times 500}} = \sqrt{\frac{R^2 116}{500}}$$





(19.) Let  $ACD$  be 45 minutes of a degree, although in the figure before us, it is many degrees.

Let  $A$  be the position of the eye of the observer, and conceive the line  $AB$ , to touch the earth at  $t$ .

Then the angle  $ACt = 4' 15''$ .

Hence,  $tCH = 40' 45''$ .

Join  $tH$ .

Now the angle made between the chord  $tH$  and the tangent  $tB$ , is measured by half the arc, therefore, the angle  $BtH = 20' 22\frac{1}{2}''$ .

The angle  $DAB$  is  $31' 20''$ , and  $BD$  is the *visible* part of the mountain, and  $BH$  is the *invisible* part. We must first compute the tangents  $At$ , and  $tB$ .

$$R : \tan. 4' 15'' :: 3956 : tA$$

$$R : \tan. 40' 45'' :: 3956 : tB$$

Log. 3956	3.597256		3.597256
Tan. $4' 15''$	<u>7.092115</u>	*tan. $40' 45''$	<u>8.073874</u>
$At$ 4.8907	0.689371	$tB$ 46.8954	1.671130
		$At$ 4.8907	
		$AB$ <u>51.7861</u>	

\* The sine or tangent of an arc exceeding  $20'$  and less than  $2^\circ$  can not be computed accurately by the common method nor by the method explained on page 288, (Plane Trigonometry.)

In the present case we may employ the familiar formula.

$$\text{Sin. } 2\alpha = 2 \text{ sin. } \alpha \cdot \text{cos. } \alpha.$$

Put

$$\alpha = (40' 45'') \div 2 = 5' 5.625''.$$

Then, we have

Log. 2	.301030	
Sin. $5' 5.625''$	7.170764	(See page 288, Trig.)
Cos. $5' 5.625''$	<u>10.000000</u>	
	<u>7.471794</u>	= sin. $10' 11.25''$ .

In the triangle  $tBH$ , we have

$$\text{Sin. } 89^\circ 39' 37\frac{1}{2}'' : 46.8954 :: \text{sin. } 20' 22\frac{1}{2}'' : BH.$$

Log. $tB$	1.671130
Sin. $20' 22\frac{1}{2}''$	7.772824
	9.443954
Sin. $89^\circ 39' 37''$	9.999992

$$BH, \text{ in parts of a mile } .27795 \quad -1.443962$$

In the triangle  $ABD$ , we have

$$\text{Sin. } 88^\circ 47' 55'' : 51.7861 :: \text{sin. } 31' 20'' : BD.$$

	Log. 51.7861	1.714213
(See note)	Sin. $31' 20''$	7.959727
		9.673940
	Sin. $88^\circ 48'$	9.999905
Visible part,	$BD, .47210$	-1.674035
Invisible part,	$BH, .27795$	
Whole mountain	$0.75005$	log. -1.875090
	Log. 5280 feet in a mile,	3.722634
Height in feet, 3960		3.597724

*2d Solution.*

We can also solve this problem by means of the triangle  $ACD$  alone.

The angle  $CAD=90^\circ 27' 5''$ .  $C=45'$ , therefore the angle  $D=88^\circ 47' 55''$ .

$$\text{Sin. } D : AC :: \text{sin. } CAD : CD.$$

Log. 2	.301030
Cos. $10' 11.25''$	9.999998
	7.772822 = sin. $20' 22.5''$ .
Log. 2	.301030
Cos. $20' 22.5''$	9.999993
	8.073845 = sin. $40' 45''$ .
Divide by cos. $40' 45''$	9.999971
	8.073874 = tan. $40' 45''$ .

$$\text{Sin. } 88^\circ 47' 55'' : AC :: \text{sin. } 90^\circ 27' 5'' : CD.$$

$$CD = \frac{AC \cdot \text{sin. } 90^\circ 27' 5''}{\text{sin. } 88^\circ 47' 55''}$$

$$CD - AC = \frac{AC \cdot \text{sin. } 90^\circ 27' 5''}{\text{sin. } 88^\circ 47' 55''} - AC$$

$$CD - AC = \frac{AC(\text{sin. } 90^\circ 27' 5'' - \text{sin. } 88^\circ 47' 55'')}{\text{sin. } 88^\circ 47' 55''} \quad (1)$$

But we have, Eq. (16), Plane Trigonometry,

$$\text{Sin. } A - \text{sin. } B = 2 \cos. \frac{1}{2}(A + B) \text{sin. } \frac{1}{2}(A - B).$$

Applying this to equation (1) above, and taking  $AC = 3956$  miles, which is sufficiently near the truth for accuracy in the required result, the operation will be as follows :

$A =$	$90^\circ 27' 5''$		
$B =$	$88^\circ 47' 55''$		
	$2)179^\circ 15' 0$	log. 2	0.301030
	$89^\circ 37' 30''$	cos.	7.815909
$A - B =$	$1^\circ 39' 10''$		
$\frac{1}{2}(A - B)$	$49' 35''$	sin.	8.159047
	Log. $AC = 3956$		3.597256
			<u>-1.873242</u>
		Sin. $88^\circ 48'$ sub.	9.999905
			<u>-1.873337</u>
		Log. of $(CD - CA)$ in parts of a mile,	3.722634
		Feet in a mile 5280, log.	<u>3.722634</u>
		Log. $(CD - CA)$ in feet, 3944.31	3.595971

We will now compute the value of  $Am$ . From (Th. 18, Book III., Geometry,) we have

$$2AC \times Am = \overline{At}^2, \text{ nearly.}$$

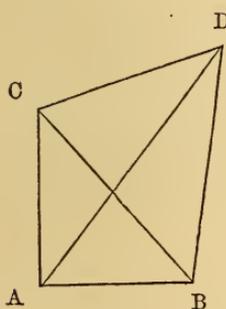
Or, 
$$Am = \frac{\overline{At}^2}{2AC}.$$

And taking again,  $AC=3956$ , we have

	Log. $\overline{At}^2$	1.378742
Log. 2	.301030	
$AC$	3.597256	
	<hr style="width: 100%;"/>	
	3.898286	3.898286
		<hr style="width: 100%;"/>
		-3.480456
	Log. 5280	3.722634
		<hr style="width: 100%;"/>
	Log. $Am$ , 15.96 ft.	1.203090

Hence,  $3944.31 + 15.96 = 3960.27$  feet =  $HD$ .

(20.) NOTE.—This problem is a singular and a very instructive one. We can not directly make use of the given side or line 500 yards, and we are forced to make a similar geometrical figure by assuming another base, and using the given angles.



The operation is as follows,

Take  $AB=100$ . Make  $BAD=52^\circ 12'$ ,  
 $CAD=41^\circ 25'$ .  $ABC=47^\circ 40'$ , and  
 $CBD=48^\circ 10'$ .

These angles make the lines intersect at  $C$  and  $D$ , and  $CD$  must be computed, and if found to be 500, then 100 is the true distance between the ships. After we have computed  $CD$ , we call it  $a$ .

Then, because this figure is similar to the true figure, we have

$$a : 500 :: 100 : AB.$$

Whence, 
$$AB = \frac{50000}{a}.$$

The angle  $ACB=38^\circ 43'$ , and  $ADB=31^\circ 58'$ .

In the triangle  $ABC$ , we have

$$\text{Sin. } 38^\circ 43' : 100 :: \text{sin. } 93^\circ 37' : BC \quad (1)$$

In the triangle  $ABD$  we have

$$\text{Sin. } 31^\circ 58' : 100 :: \text{sin. } 52^\circ 12' : BD \quad (2)$$

Log. 100 sin. $93^\circ 37'$	11.999134
Log. sin. $38^\circ 43'$	9.796206
Log. $BC$ , 159.561	2.202928

Log. 100 sin. $52^\circ 12'$	11.897712
Log. sin. $31^\circ 58'$	9.723805
Log. $BD$ , 149.247	2.173907

$CB + BD = 308.808.$      $CB - DB = 10.314.$

In the triangle  $CBD$ , we have

$$308.808 : 10.314 :: \tan. 65^\circ 55' : \tan. \frac{1}{2}(D-C).$$

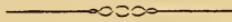
Log. 10.314	1.013427
Log. tan. $65^\circ 55'$	10.349719
	11.363146
Log. 308.808	2.489689
Tan. $\frac{1}{2}(D-C) = 4^\circ 16' 24''$	8.873457
$\frac{1}{2}(D+C) = 65^\circ 55'$	
$D = 70^\circ 11' 24''$	

$$\text{Sin. } 70^\circ 11' 24'' : 159.561 :: \text{sin. } 48^\circ 10' : CD.$$

Log. 159.561	2.202928
Log. sin. $48^\circ 10'$	9.872208
	12.075136
Log. sin. $70^\circ 11' 24''$	9.973507
Log. $CD = 126.366$	2.101629
Log. 50000	4.698970
Log. $AB$ , 395.68	2.597341

} sub. log.  $CD$ .

To obtain the true values of  $CB$ ,  $BD$ , &c., we must multiply the results in this computation by 3.9568. That is, the sides of the true figure are very nearly equal to 4 times the sides of this computed figure.



## SPHERICAL TRIGONOMETRY.

(Page 310.)

To solve right angled spherical triangles, the student will find it most convenient to apply *Napier's Circular points*.

Every triangle consists of *three* sides and two angles, besides the right angle, or *five parts*.

Either one of these parts taken at pleasure may be called a *middle part*. Then there will be two *adjacent parts*, and two *opposite parts*.

Then we can easily remember, that

1st. *Radius into the sine of the middle part, is equal to the product of the tangents of the adjacent parts.*

2d. *Radius into the sine of the middle part, is equal to the product of the cosines of the opposite parts.*

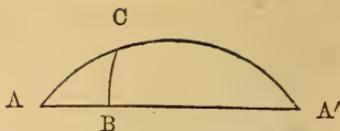
The parts are the two sides, *the complements of the hypotenuse*, and the *complement of the oblique angles*.

We can remember these rules by this consideration, that each rule expresses an equation.

The first member of each of these equations is *radius into some sine*, the second member is the product of *two tangents* or *two cosines*.

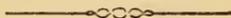
The student must rely on his own judgment, in selecting a part to be called the *middle part*.

Let  $ABC$  represent any right angled spherical triangle, the right angle at  $B$ .



If the hypotenuse is greater than  $90^\circ$ , then the triangle will be represented by  $A'BC$ , and in that case one side will be greater than  $90^\circ$ , and the angle opposite to that side greater than  $90^\circ$ .

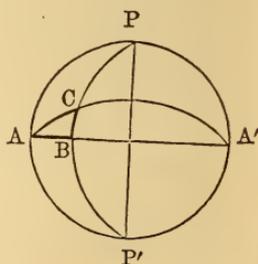
For the sake of perspicuity, we recommend the natural construction of each triangle presented for solution. This course will banish all doubts from the mind of the student as to what the results should be, &c. The solution of one triangle will in all cases be the solution of a whole hemisphere of triangles, as we are about to show by the following examples.



### RIGHT ANGLED SPHERICAL TRIGONOMETRY.

(Page 356.)

In a right angled spherical triangle  $ABC$  (right angles always at  $B$ ), given  $AB=118^\circ 21' 4''$ , and the angle  $A=23^\circ 40' 12''$ .



REMARK.—This triangle is represented in the adjoining figure by the triangle  $A'BC$ , but we always operate on the triangle  $ABC$  of the figure, the one whose parts are each less than  $90^\circ$ .

Whence,  $AB=180^\circ-118^\circ 21' 4''=61^\circ 38' 56''=c$ .

To this triangle we apply equation (16), page 335, Text-book), observing that

$$\text{Sin. } A' = \cos. A \text{ and } \tan. b' = \cot. b.$$

Whence,  $R.\cos. A = \tan. 61^\circ 38' 56'' \cot. b.$

Or.	$\text{Cot. } b = \frac{R.\cos. A}{\tan. c}$	19.961835
		10.267932
	$AC \cot. 63^\circ 42' 5''$	9.693903
	$180^\circ$	
	$A'C \quad 116^\circ 17' 55''$	

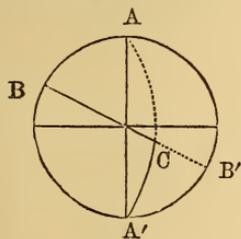
For the angle  $c$ .

As  $\text{Sin. } 63^\circ 42' 5'' : R :: \text{sin. } 61^\circ 38' 56'' : \text{sin. } C.$

$R.\text{sin. } c =$	19.944510
$\text{Sin. } b$	9.952549
$C = 79^\circ 0' 34''$	9.991961

Or,  $100^\circ 59' 26''.$  *Ans.*

(2.) In a spherical  $\triangle ABC$ , given  $AB = 53^\circ 14' 20''$ , and the angle  $A = 91^\circ 25' 53''$ .



Here the proposed triangle is  $ABC$ , but we shall operate on its supplemental triangle  $A'B'C$ .  $AB = A'B$ . The angle  $CA'B' = 90^\circ - (1^\circ 25' 53'') = 88^\circ 34' 7''$ .

Eq. (16.)  $R. \cos. 88^\circ 34' 7'' = \tan. 53^\circ 14' 20''. \cot. A'C.$

Or,	$\text{Cot. } A'C = \frac{R.\text{sin. } 1^\circ 25' 53''}{\text{Tan. } 53^\circ 14' 20''}$	18.397585
		10.126658
	$\text{Cot. } A'C \quad 88^\circ 55' 51''$	8.270927
	$180$	
	$AC = 91^\circ 4' 9''.$	<i>Ans.</i>

For  $OB$ , we have  $R : \text{sin. } A'C :: \text{sin. } A' : \text{sin. } B'C.$

Whence,  $\sin. B'C = \frac{\sin. 88^\circ 55' 51'', \sin. 88^\circ 34' 7''}{R}$

Sin.  $88^\circ 55' 51''$  9.999925

Sin.  $88^\circ 34' 7''$  9.999864

$B'C, \sin. 88^\circ 12' 50''$  9.999789

Sup. =  $BC, 91^\circ 47' 10''$ . Ans.

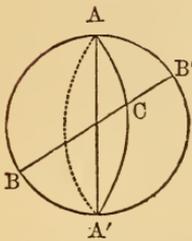
Sin.  $A'C : \sin. 90^\circ :: \sin. AB : \sin. C$ .

$R.\sin. AB$  19.903707

Sin.  $AC = \sin. A'C 88^\circ 55' 51''$  9.999925

Ans. Sin.  $C = 53^\circ 15' 8''$  9.903782

(3.) In this example,



Let  $AB = 102^\circ 50' 25''$ , and the angle  $BAC = 113^\circ 14' 37''$ , but we shall operate on the supplemental triangle  $AB'C$ ;  $AB' = 77^\circ 9' 35''$ , and  $CAB' = 66^\circ 45' 23''$ .

Eq. (16), calling  $A$  the middle part,  
 $R.\cos. 66^\circ 45' 23'' = \cot. AC.\tan. 77^\circ 9' 35''$ .

$\cot. AC = \frac{R.\cos. 66^\circ, \&c.}{\tan. AB}$  19.596202  
 10.642190

Ans.  $\cot. AC = \cot. 84^\circ 51' 36''$  8.954012

For  $B'C$ , we have

$R.\cos. AC = \cos. AB' . \cos. B'C$ .

$R.\cos. AC, 84^\circ 51' 36''$  18.952258

$\cos. AB' 77^\circ 9' 35''$  9.346811

$\cos. B'C, 66^\circ 13' 33''$  9.605447

And Supplement,  $BC, 113^\circ 46' 27''$ . Ans.

For the angle  $C$ , we have

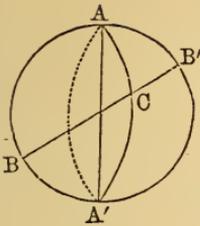
Sin.  $AC : R :: \sin. AB' : \sin. C$ .

Log.  $R.\sin. AB' 77^\circ 9' 35''$  19.989002

Sin.  $AC, 84^\circ 51' 36''$  9.998250

Sin.  $ACB', \sin. 78^\circ 13' 4''$  9.990752

Supple.  $ACB = 101^\circ 46' 56''$ . Ans.



(4.) Let  $AB'C$  in the last cut represent the triangle to be solved,  $AB' = 48^\circ 24' 16''$ , and  $B'C, 59^\circ 38' 27''$ .

Whence,

$$R.\cos. AC = \cos. 48^\circ 24' 16'' . \cos. 59^\circ 38' 27''.$$

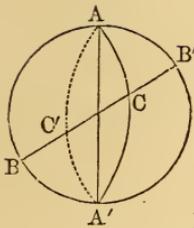
Cos. $48^\circ 24' 16''$	9.822082
Cos. $59^\circ 38' 27''$	9.703652
Cos. $AC, \cos. 70^\circ 23' 42''$	<u>9.525734.</u> <i>Ans.</i>

For the angles, we have the following proportions.

$$\begin{aligned} \text{Sin. } 70^\circ 23' 42'' : \text{sin. } 90^\circ &:: \text{sin. } 48^\circ 24' 16'' : \text{sin. } C \\ &:: \text{sin. } 59^\circ 38' 27'' : \text{sin. } A. \end{aligned}$$

Log. $R.\text{sin. } 48^\circ 24' 16''$	19.873814
Sin. $70^\circ 23' 42''$	9.974064
Angle $C, 52^\circ 32' 56''$	<u>9.899750</u>

Log. $R.\text{sin. } 59^\circ 38' 27''$	19.935948
sin. $70^\circ 23' 42''$	9.974064
Angle $A, 66^\circ 20' 40''$	<u>9.961884.</u> <i>Ans.</i>



(5.) In this example,  $AB$  is  $151^\circ 23' 9''$ , and  $BC', 16^\circ 35' 14''$ .

We operate on the supplemental triangle  $A'BC'$ . Whence

$$A'B = 28^\circ 36' 51''. \text{ And,}$$

$$R.\cos. A'C' = \cos. 28^\circ 36' 51'' . \cos. 16^\circ 35' 14''.$$

Cos. $28^\circ 36' 51''$	9.943427
Cos. $16^\circ 35' 14''$	9.981540
$A'C', \cos. 32^\circ 43' 9''$	<u>9.924967</u>
$AC', 147^\circ 16' 51''.$	<i>Ans.</i>

For the angle  $C$ ,

$$\text{Sin. } A'C' : \text{sin. } 90^\circ :: \text{sin. } A'B : \text{sin. } A'C'B.$$

$$\text{Log. } R.\text{sin. } 28^\circ 36' 51'' \quad 19.680253$$

$$\text{Sin. } A'C'. 32^\circ 43' 9'' \quad 9.732814$$

$$\text{sin. } A'C'B, 62^\circ 22' 35'' \quad \underline{9.947439}$$

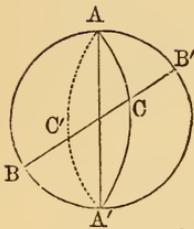
Its supplement, or  $AC'B=117^\circ 37' 25''$

For the angle  $A$ , we have

$$R.\text{sin. } 16^\circ 35' 14'' \quad 19.455568$$

$$\text{sin. } 32^\circ 43' 9'' \quad 9.732813$$

$$\text{Sin. } A=31^\circ 52' 49'' \quad \underline{9.722755. \text{ Ans.}}$$



(6.) Here  $A'B, 73^\circ 4' 31''$ ,  $A'C', 86^\circ 12' 15''$ .

Required the other side and angles.

$$\text{Sin. } A'C' : \text{sin. } 90^\circ :: \text{sin. } A'B : \text{sin. } BC'A'.$$

$$\text{Log. } R.\text{sin. } 73^\circ 4' 31'' \quad 19.980771$$

$$A'C', \text{ sin. } 86^\circ 12' 15'' \quad 9.999046$$

$$\text{Sin. } BC'A', \text{ sin. } 73^\circ 29' 40'' \quad \underline{9.981725. \text{ Ans.}}$$

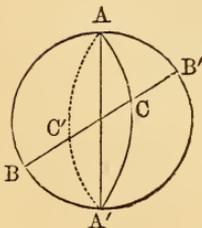
To obtain  $BC'$ , we apply one of the equations in circular parts.

$$R.\text{cos. } C' = \tan BC'.\text{cot. } A'C'$$

$$\text{Tan. } BC' = \frac{R.\text{cos. } C'}{\text{cot. } A'C'} \quad 19.453484$$

$$8.821819$$

$$BC' 76^\circ 51' 20'' \quad \underline{10.631665. \text{ Ans.}}$$



(7.) Let  $AB'C'$  represent the proposed triangle.

Then,  $AB'=47^\circ 26' 35''$ , and  $AC'=118^\circ 32' 12''$ .

We operate on the opposite supplemental triangle,  $A'BC'$ .

Now,  $A'B = 47^\circ 26' 35''$ .  $A'C' = 61^\circ 27' 48''$ .

$$\text{Sin. } 61^\circ 27' 48'' : R. :: \text{sin. } 47^\circ 26' 35'' : \text{sin. } C'$$

$$\text{Sin. } C' = \frac{R. \text{sin. } 47^\circ 26' 35''}{\text{sin. } 61^\circ 27' 48''} \quad \begin{array}{r} 19.867235 \\ \hline 9.943748 \end{array}$$

$$C' = AC'B' = 56^\circ 58' 44'' \quad \begin{array}{r} \hline 9.923487. \text{ Ans.} \end{array}$$

Again, we have

$$R. \cos. 56^\circ 58' 44'' = \cot. 61^\circ 27' 48'' . \tan. BC'$$

Whence,

$$\text{Tan. } BC' = \frac{R. \cos. 56^\circ 58' 44''}{\cot. 61^\circ 27' 48''} \quad \begin{array}{r} 19.736355 \\ \hline 9.735427 \end{array}$$

$$BC', \tan. 45^\circ 3' 40'' \quad \begin{array}{r} \hline 10.000928 \end{array}$$

$180^\circ$

$$B'C' \quad \begin{array}{r} \hline 134^\circ 56' 20''. \text{ Ans.} \end{array}$$

To find the angle  $A$ , we have

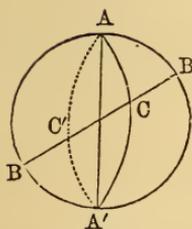
$$\text{Sin. } A'C' : \text{sin. } 90^\circ :: \text{sin. } BC' : \text{sin. } BA'C'.$$

$$\text{Log. } R. \text{sin. } BC', 45^\circ 3' 40'' \quad 19.849948$$

$$\text{Sin. } A'C', 61^\circ 27' 48'' \quad \begin{array}{r} \hline 9.943748 \end{array}$$

$$\text{Sin. } BA'C', 53^\circ 40' 58'' \quad \begin{array}{r} \hline 9.906200 \end{array}$$

$$\text{Supplement } A, 126^\circ 19' 2'' \quad \text{Ans.}$$



(8.) Let  $AB'C'$  represent the triangle,  $AB' = 40^\circ 18' 23''$ , and  $AC' = 100^\circ 3' 7''$ . Required the other side and the oblique angles.

We operate as before on the supplemental triangle  $A'BC'$ .  $AB' = A'B = 40^\circ 18' 23''$ .  $A'C' = 79^\circ 56' 53''$ .

1st. For the angle  $C'$ , we have

$$\text{Sin. } A'C' : \text{sin. } 90^\circ :: \text{sin. } A'B : \text{sin. } C'.$$

$$\text{Log. } R. \text{sin. } 40^\circ 18' 23'' \quad 19.810821$$

$$A'C', \text{sin. } 79^\circ 56' 53'' \quad \begin{array}{r} \hline 9.993282 \end{array}$$

$$C', \text{sin. } 41^\circ 4' 6'' \quad \begin{array}{r} \hline 9.817539. \text{ Ans.} \end{array}$$

To find  $BC'$ , take the angle  $C'$  for a middle part, then by Napier's Circular Parts, we have

$$\text{Tan. } BC', \text{ cot. } A'C' = R.\text{cos. } 41^\circ 4' 6''.$$

$$\text{Log. } R.\text{cos. } 41^\circ 4' 6'' \quad 19.877329$$

$$A'C', \text{ cot. } 79^\circ 56' 53'' \quad 9.248616$$

$$BC', \text{ tan. } 76^\circ 46' 8'' \quad \underline{10.628713}$$

$$\text{Supplement } B'C', 103^\circ 13' 52''. \text{ Ans.}$$

For the angle  $A$ , we have

$$\text{Sin. } A'C' : \text{sin. } 90^\circ :: \text{sin. } BC' : \text{sin. } A'.$$

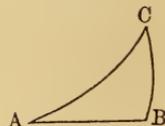
$$\text{Log. } R.\text{sin. } 76^\circ 46' 8'' \quad 19.988316$$

$$\text{sin. } 79^\circ 56' 53'' \quad 9.993282$$

$$A', \text{ sin. } 81^\circ 21' 7'' \quad \underline{9.995034}$$

$$\text{Supplement } C'AB' = 98^\circ 38' 53''. \text{ Ans.}$$

(9.) In the right angled spherical triangle  $ABC$ , given  $AC$ ,  $61^\circ 3' 22''$ , and the angle  $A$ ,  $49^\circ 28' 12''$ , to find the other parts.



As,  $\text{Sin. } 90^\circ : \text{sin. } AC, 61^\circ 3' 22'' :: \text{sin. } A, 49^\circ 28' 12'' : \text{sin. } BC.$

$$AC, \text{ sin. } 61^\circ 3' 22'' \quad 9.942054$$

$$A, \text{ sin. } 49^\circ 28' 12'' \quad 9.880852$$

$$BC, \text{ sin. } 41^\circ 41' 32'' \quad \underline{9.822906. \text{ Ans.}}$$

For  $AB$ , we have the equation,

$$R.\text{cos. } AC = \text{cos. } AB.\text{cos. } BC.$$

$$\text{Cos. } AB = \frac{R.\text{cos. } AC}{\text{cos. } BC} \quad 19.684803$$

$$\underline{9.873162}$$

$$\text{Cos. } 49^\circ 36' 6'' \quad 9.811641. \text{ Ans.}$$

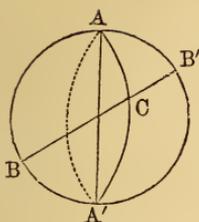
For the angle  $C$ , we have

$\text{Sin. } AC, 61^\circ 3' 22'' : R. :: \text{sin. } AB, 49^\circ 36' 6'' : \text{sin. } C.$

$$R.\text{sin. } AB \quad 19.881703$$

$$\text{sin. } AC \quad 9.942054$$

$$C, \text{ sin. } 60^\circ 29' 20'' \quad \underline{9.939649. \text{ Ans.}}$$



(10.) Here we have given one side of a right angled spherical triangle, and its opposite angle, to determine the other parts of the triangle,  $AB'C$ .  $AB' = 29^\circ 12' 50''$ , and  $C = 37^\circ 26' 21''$ .

In such cases the answers are said to be *ambiguous*, for the data give us no indication of which triangle is intended,  $AB'C$  or  $A'BC$ . Because,  $AB' = A'B = 29^\circ 12' 50''$ , and  $A'CB = ACB' = 37^\circ 26' 21''$ .

To find  $AC$ , we have

$$\text{Sin. } AB', 29^\circ 12' 50'' : \text{sin. } 37^\circ 26' 21'' :: \text{sin. } AC : R.$$

$$\text{Sin. } AC = \frac{R \cdot \text{sin. } 29^\circ 12' 50''}{\text{sin. } 37^\circ 26' 21''} \quad \begin{array}{r} 19.688483 \\ 9.783846 \\ \hline 9.904637. \text{ Ans.} \end{array}$$

Supplement  $A'C$ ,  $126^\circ 35' 47''$ .

To find  $B'C$ .

$$R \cdot \cos. AC = \cos. AB' \cos. B'C.$$

$$\text{Cos. } B'C = \frac{R \cdot \cos. 53^\circ 24' 13''}{\cos. 29^\circ 12' 50''} \quad \begin{array}{r} 19.775372 \\ 9.940916 \\ \hline 9.834456. \text{ Ans.} \end{array}$$

Or its supplement.

For the angle  $A$ , we have

$$\text{Sin. } 53^\circ 24' 13'' : R. :: \text{sin. } 46^\circ 55' 2'' : \text{sin. } A.$$

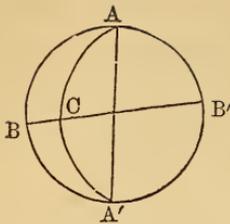
$$\text{Log. } R \cdot \text{sin. } 46^\circ 55' 2'' \quad 19.863542$$

$$\text{sin. } 53^\circ 24' 13'' \quad 9.904637$$

$$\text{Sin. } A, 65^\circ 27' 57'' \quad \hline 9.958905. \text{ Ans.}$$

Or its supplement.

(11.) This example may require the solution of the triangle  $ABC$ , in adjoining figure, or of the triangle  $A'B'C$ .



But the solution of either of these supplemental triangles is effected by the triangle  $A'BC$ , which is a common supplemental triangle. We have

$$AB = A'B' = 100^\circ 10' 3''.$$

$$ACB = A'CB' = 90^\circ 14' 20''.$$

Also,  $A'B = 79^\circ 49' 57''$ , and  $A'CB = 89^\circ 45' 40''$ .

$$\text{Sin. } C : \text{sin. } A'B :: \text{sin. } 90^\circ : \text{sin. } A'C.$$

Or,  $\text{Sin. } 89^\circ 45' 40'' : \text{sin. } 79^\circ 49' 57'' :: R. : \text{sin. } A'C.$

Whence,

$$\text{Sin. } A'C = \frac{R. \text{sin. } 79^\circ 49' 57''}{\text{sin. } 89^\circ 45' 40''} \quad \begin{array}{r} 19.993126 \\ 9.999996 \end{array}$$

$$A'C, \text{ sin. } 79^\circ 50' 8'' \quad \begin{array}{r} 9.993130 \\ 180^\circ \end{array}$$

$$AC, \quad \frac{100^\circ 9' 52''}{180^\circ} \text{ Ans.}$$

By Eq. (20), Napier's Circular Parts, we have

$$R. \cos. C = \cos. 79^\circ 49' 57'' \cdot \text{sin. } A.$$

$$\cos. C = \cos. 89^\circ 45' 40'' = \text{sin. } 0^\circ 14' 20'' = \text{sin. } 860'' = 860 \text{ sin. } 1''.$$

$$\text{Log. sin. } 1'' \quad 4.685575 \text{ (See page 288 Text-book.)}$$

$$\text{Log. } R. \text{ 860} \quad 12.934498$$

$$\text{Log. } R. \cos. C \quad 17.620073$$

$$\text{Log. cos. } 79^\circ 49' 57'' \quad 9.246810$$

$$BAC = \text{sin. } 1^\circ 21' 12'' \quad 8.373263 \text{ Ans.}$$

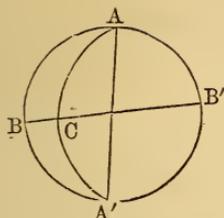
For  $BC$ , we have

$$R. : \text{sin. } A'C :: \text{sin. } A : \text{sin. } BC.$$

$$\text{Sin. } A'C, 79^\circ 50' 8'' \quad 9.993130$$

$$\text{Sin. } A, 1^\circ 21' 12'' \quad 8.373263$$

$$BC, \text{ sin. } 1^\circ 19' 55'' \quad \frac{8.366393}{8.366393} \text{ Ans.}$$



(12.) This example may be represented by the triangle  $A'BC$ , or its supplemental triangle  $AB'C$ , in the figure.

In the triangle  $A'BC$ , we have

$$\text{Sin. } C, 61^\circ 2' 15'' : \text{sin. } A'B, 54^\circ 21' 35'' :: [R. : \text{sin. } A'C.$$

$$\begin{array}{r} \text{Sin. } A'C = \frac{R \cdot \text{sin. } A'B}{\text{sin. } 61^\circ 2' 15''} \\ \hline A'C, 68^\circ 15' 26'' \\ \hline 180^\circ \end{array}$$

$$AC, 111^\circ 44' 34''. \text{ Ans.}$$

For  $BC$ , we have  $R \cdot \text{sin. } BC = \tan. A'B \cdot \cot. C$ .

$$\begin{array}{r} \text{Tan. } A'B \\ \text{Cot. } C \end{array} \quad \begin{array}{r} 10.144485 \\ 9.743081 \\ \hline \end{array}$$

$$\begin{array}{r} \text{Sin. } BC, 50^\circ 31' 32'' \\ \hline 180^\circ \end{array}$$

$$B'C = 129^\circ 28' 28''$$

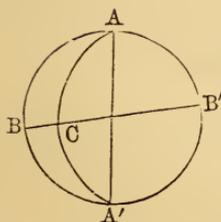
For the angle  $BA'C$ ,

$R \cdot \cos. BA'C = \tan. A'B \cdot \cot. A'C$ .

$$\begin{array}{r} \text{Tan. } A'B \\ \text{Cot. } A'C \end{array} \quad \begin{array}{r} 10.144485 \\ 9.600770 \\ \hline \end{array}$$

$$\begin{array}{r} \text{Cos. } BAC = \cos. BA'C, 56^\circ 12' 16'' \\ \hline 180^\circ \end{array}$$

$$B'AC = 123^\circ 47' 44''$$



(13.) This example may be made to correspond to the triangle  $ACB$ , or to the triangle  $A'B'C$ , because the opposite angles  $ACB, A'CB'$  are equal, and  $AB = A'B$ .

But we shall operate on the triangle  $A'BC$ , which is supplemental to each of the other two.

Because  $AB=121^{\circ} 26' 25''$ ,  $A'B=58^{\circ} 33' 35''$ .

Because  $ACB=111^{\circ} 14' 37''$ ,  $A'CB=68^{\circ} 45' 23''$ .

For the side  $A'C$ , we have

$$\text{Sin. } C : \text{sin. } A'B :: R. : \text{sin. } A'C.$$

$$\text{Log. } R.\text{sin. } 58^{\circ} 33' 35'' \qquad 19.931043$$

$$\text{Sin. } C, 68^{\circ} 45' 23'' \qquad 9.969439$$

$$A'C, \text{sin. } 66^{\circ} 15' 38'' \qquad \underline{9.961604}$$

Supplement, or  $AC$ ,  $113^{\circ} 44' 22''$ . *Ans.*

For the angle  $A$ , or  $BAC$ , we have

$$\text{Tan. } A'B.\text{cot. } A'C=R.\text{cos. } A'=R.\text{cos. } A.$$

Or,  $\text{Cos. } A = \text{tan. } 58^{\circ} 33' 35''.\text{cot. } 66^{\circ} 15' 38''$ .

$$\text{Tan. } 58^{\circ} 33' 35'' \qquad 10.213698$$

$$\text{Cot. } 66^{\circ} 15' 38'' \qquad 9.643246$$

$$A, \text{cos. } 43^{\circ} 59' 55'' \qquad \underline{9.856944}$$

Or, supplement,  $B'A'C$ ,  $136^{\circ} 0' 5''$ . *Ans.*

For the side  $BC$ , we have

$$R. : \text{sin. } A'C :: \text{sin. } A' : \text{sin. } BC.$$

$$\text{Sin. } A'C \qquad 9.961604$$

$$\text{Sin. } A' \qquad 9.841760$$

$$BC, \text{sin. } 39^{\circ} 29' 3'' \qquad \underline{9.803364}$$

Supplement, or  $B'C$ ,  $140^{\circ} 30' 57''$   $AC$ .

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### QUADRANTAL TRIANGLES.

Quadrantal spherical triangles have one side equal to 90 degrees, and all such triangles can be solved by right angled spherical trigonometry, as illustrated in the Text-book.

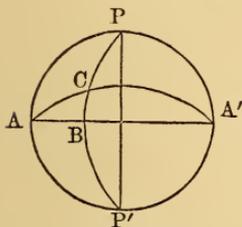
The following are not solved in the Text-book.

PRACTICAL PROBLEMS.

(Page 361.)

(1.) In a quadrantal triangle, given the quadrantal side,  $90^\circ$ , a side adjacent,  $67^\circ 3'$ , and the included angle,  $49^\circ 18'$  to find the other parts.

*Ans.* { The remaining side is  $53^\circ 5' 44''$ ; the angle opposite the quadrantal side,  $108^\circ 32' 29''$ ; and the remaining angle,  $60^\circ 48' 54''$ .



The triangle corresponding to this example is represented by  $APC$ .  $AP=90^\circ$ ,  $PC=67^\circ 3'$ , and the angle at  $P=49^\circ 18'$ .

We operate upon the triangle  $ABC$ , having  $AB=P=49^\circ 18'$ , and  $BC=90^\circ - (67^\circ 3') = 22^\circ 57'$ .

For  $AC$ ,

$$R.\cos. AC = \cos. 49^\circ 18'. \cos. 22^\circ 57'$$

$$\cos. 49^\circ 18' \qquad \qquad \qquad 9.814313$$

$$\cos. 22^\circ 57' \qquad \qquad \qquad 9.964187$$

$$AC, \cos. 53^\circ 5' 44'' \qquad \qquad \qquad \underline{9.778500.} \quad \text{Ans.}$$

For the angles  $A$  and  $C$ , we have the proportions

$$\sin. AC : R. :: \sin. AB : \sin. C$$

$$:: \sin. BC : \sin. A.$$

For  $C$ .

For  $A$ .

$$\text{Log. } R.\sin. 49^\circ 18' \quad 19.879746 \quad \text{Log. } R.\sin. 22^\circ 57' \quad 19.590984$$

$$\sin. 53^\circ 5' 44'' \quad 9.902894 \qquad \qquad \sin. AC \quad 9.902894$$

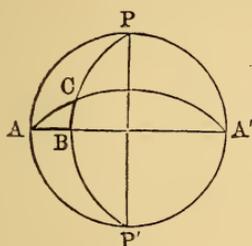
$$C, 71^\circ 27' 31'' \quad \underline{9.976852} \qquad \qquad A, 29^\circ 11' 6'' \quad \underline{9.688090}$$

$$\text{Sup. } ACP, 108^\circ 32' 29''. \text{ Ans.} \quad \text{Com. } PAC, 60^\circ 48' 54'' \text{ Ans.}$$



(3.) In a quadrantal triangle given the quadrantal side  $90^\circ$ , and the two adjacent angles, one  $69^\circ 13' 46''$ , the other  $72^\circ 12' 4''$ , to find the other parts.

*Ans.* } One of the remaining sides is  $70^\circ 8' 39''$ , the other  
is  $73^\circ 17' 29''$ , and the angle opposite the quadrantal side is  $96^\circ 13' 23''$ .



The triangle is represented by  $PAC$ . Taking the angle  $P=69^\circ 13' 46''$ , then  $AB=69^\circ 13' 46''$ , and the angle  $BAC$  must equal  $90^\circ - (72^\circ 12' 4'')$ , or,  $17^\circ 47' 56'' = A$ .

Now, by taking the angle  $A$  for a middle part, we have the equation,

$$\text{Cot. } AC \cdot \tan. AB = R \cdot \cos. A.$$

Whence,

$$\begin{array}{r} \text{Cot. } AC = \frac{R \cdot \cos. A, 17^\circ 47' 56''}{\tan. 69^\circ 13' 46''} \quad \begin{array}{r} 19.978698 \\ 10.421044 \\ \hline 9.557654. \end{array} \end{array} \quad \text{Ans.}$$

For the side  $BC$ , of the triangle  $ABC$ , we have

$$\begin{array}{r} R. : \sin. AC :: \sin. A : \sin. BC. \\ \sin. AC, 70^\circ 8' 39'' \quad \quad \quad 9.973382 \\ \sin. A, 17^\circ 47' 56'' \quad \quad \quad 9.485262 \\ \hline BC, \sin. 16^\circ 42' 31'' \quad \quad \quad 9.458644 \\ \text{Complement, } PC, 73^\circ 17' 29''. \quad \text{Ans.} \end{array}$$

For the angle  $C$ , of the triangle  $ABC$ , we have

$$\begin{array}{r} R \cdot \cos. AC = \cot. A \cdot \cot. C. \\ R \cdot \cos. AC \quad \quad \quad 19.531037 \\ \cot. A \quad \quad \quad 10.493436 \\ \hline \cot. C, 83^\circ 46' 37'' \quad \quad \quad 9.037601 \\ \text{Supplement, } PCA = 96^\circ 13' 23''. \quad \text{Ans.} \end{array}$$

(4.) In a quadrantal triangle, given the quadrantal side,  $90^\circ$ , one adjacent side,  $86^\circ 14' 40''$ , and the angle opposite to that side,  $37^\circ 12' 20''$ , to find the other parts.

*Ans.* { The remaining side is  $4^\circ 43' 2''$ ; the angle opposite  
 $2^\circ 51' 23''$ ; and the angle opposite the quadrantal  
side,  $142^\circ 42' 3''$ .

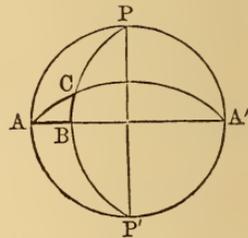
The triangle is represented by  $PAC$ ,  
 $PC=86^\circ 14' 40''$ , and  $PAC=37^\circ 12' 20''$ .

Whence, the angle  $A$ , of the triangle  
 $ABC=52^\circ 47' 40''$ , and  $BC$ ,  $3^\circ 45' 20''$ .

For  $AC$ , we have

$\text{Sin. } A : \text{sin. } BC :: R. : \text{sin. } AC.$

Log. $R.\text{sin. } BC$ , $3^\circ 45' 20''$	18.816240
$\text{sin. } A$ , $52^\circ 47' 40''$	9.901170
$\text{Sin. } AC$ , $4^\circ 43' 2''$	<u>8.915070.</u> <i>Ans.</i>



For  $AB$ , we have

$R.\text{cos. } A = \text{cot. } AC. \text{tan. } AB.$

$R.\text{cos. } 52^\circ 47' 40''$	19.781523
$\text{Cot. } 4^\circ 43' 2''$	11.083454
$\text{Tan. } AB$ , $2^\circ 51' 23''$	<u>8.698069.</u> <i>Ans.</i>

For the angle  $C$ , of the triangle  $ABC$ ,

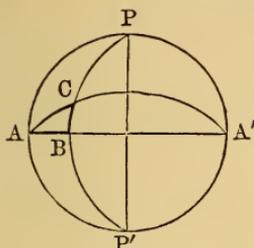
$R.\text{cos. } AC = \text{cot. } A. \text{cot. } C.$

$R.\text{cos. } 4^\circ 43' 2''$	19.998527
$\text{cot. } 52^\circ 47' 40''$	9.880353
$\text{Cot. } C$ , $37^\circ 17' 57''$	<u>10.118174</u>
Supplement, $PCA=142^\circ 42' 3''$ .	<i>Ans.</i>

(5.) In a quadrantal triangle, given the quadrantal side,  $90^\circ$ , and the other two sides, one  $118^\circ 32' 16''$ , the

other  $67^\circ 48' 40''$ , to find the other parts—the three angles.

*Ans.* { The angles are  $64^\circ 32' 21''$ ,  $121^\circ 3' 40''$ , and  $77^\circ 11' 6''$ ; the greater angle opposite the greater side, of course.



This problem requires the solution of the triangle  $A'PC$ , or  $P'AC$ , each one may by hypothesis correspond with the data. That is,  $A'C$  or  $P'C=118^\circ 32' 16''$ , and  $PC$  or  $AC=67^\circ 48' 40''$ .

We will take  $P'AC$ . Then,  $BC=118^\circ 32' 16''$ , less  $90^\circ$ .

That is,  $BC=28^\circ 32' 16''$ .

In the triangle  $ABC$ , we have  $AC$  and  $BC$ .

To find the angle  $A$ , or  $CAB$ , we have

$$\text{Sin. } AC, 67^\circ 48' 40'' : R. :: \text{sin. } 28^\circ 32' 16'' : \text{sin. } A.$$

$$\text{Log. } R.\text{sin. } 28^\circ 32' 16'' \qquad 19.679191$$

$$\text{sin. } 67^\circ 48' 40'' \qquad \underline{9.966585}$$

$$\text{Sin. } A, 31^\circ 3' 40'' \qquad \underline{9.712606}$$

$$\text{Add } 90^\circ \text{ and } P'AC=121^\circ 3' 40''. \text{ Ans.}$$

The side  $AB$ , is the measure of the angles  $P$  and  $P'$ , and the angle  $C$  is also an angle in the triangle  $P'AC$ .

For  $AB$ ,  $R. : \text{cos. } AB :: \text{cos. } BC : \text{cos. } AC$ .

$$\text{Cos. } AB = \frac{R.\text{cos. } AC, 67^\circ 48' 40''}{\text{cos. } BC, 28^\circ 32' 16''} \qquad 19.577102$$

$$\qquad \qquad \qquad \underline{9.943743}$$

$$\text{Angle } P'=AB, 64^\circ 32' 21'' \qquad \underline{9.633359. \text{ Ans.}}$$

For the angle  $C$ , we have

$$\text{Sin. } AC : R. :: \text{sin. } AB : \text{sin. } C.$$

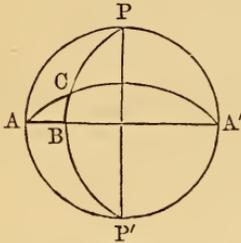
$$\text{Log. } R.\text{sin. } AB, 64^\circ 32' 21'' \qquad 19.955630$$

$$\text{Log. } R.\text{sin. } AC, 67^\circ 48' 40'' \qquad \underline{9.966585}$$

$$\text{sin. } C, 77^\circ 11' 6'' \qquad \underline{9.989045. \text{ Ans.}}$$

(6.) In a quadrantal triangle, given the quadrantal side,  $90^\circ$ , the angle opposite,  $104^\circ 41' 17''$ , and one adjacent side,  $73^\circ 21' 6''$ , to find the other parts.

*Ans.* { Remaining side,  $49^\circ 42' 16''$ ; remaining angles,  $47^\circ 32' 38''$ , and  $67^\circ 56' 13''$ .



This example refers to the triangle  $APC$ , because the given angle is greater than  $90^\circ$ . We must operate on the triangle  $ABC$ .

The angle  $ACB = 180 - (104^\circ 41' 17'')$   
 $= 75^\circ 18' 43''$ , and  $AC, 73^\circ 21' 6''$ .

For  $AB$ , we have

$$R. : \sin. AC :: \sin. C : \sin. AB.$$

Whence,  $\text{Sin. } AB = \sin. AC \cdot \sin. C.$

Sin. $75^\circ 18' 43''$	9.985571	
$AC, \sin. 73^\circ 21' 6''$	9.981402	
$APC = AB, \sin. AB, 67^\circ 56' 13''$	9.966973.	<i>Ans.</i>

For the side  $BC$ , we have

$$R. : \cos. AB :: \cos. BC : \cos. AC.$$

$$\text{Cos. } BC = \frac{R \cdot \cos. AC}{\cos. AB} \quad \begin{array}{r} 19.457120 \\ \hline 9.574757 \end{array}$$

$$\text{Cos. } BC, 40^\circ 17' 44'' \quad \begin{array}{r} 8.882363 \end{array}$$

Complement,  $PC = 49^\circ 42' 16''$ . *Ans.*

For the angle  $A$ , we have

$$\text{Sin. } AC : R. :: \sin. BC : \sin. A.$$

$$\text{Log. } R \cdot \sin. BC \quad \begin{array}{r} 19.810723 \\ \hline 9.981402 \end{array}$$

$$\sin. 73^\circ 21' 6'' \quad \begin{array}{r} 9.981402 \end{array}$$

$$\text{Sin. } A, 42^\circ 27' 22'' \quad \begin{array}{r} 9.829321 \end{array}$$

Complement,  $PAC = 47^\circ 32' 38''$ . *Ans.*

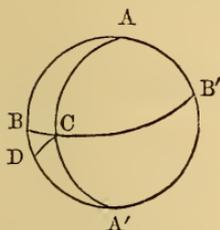
## OBLIQUE ANGLED SPHERICAL TRIGONOMETRY.

## PRACTICAL EXAMPLES.

(Page 367.)

NOTE.—Here, as in Plane Trigonometry, the sides are represented by  $a$ ,  $b$ ,  $c$ , and the angles opposite by  $A$ ,  $B$ ,  $C$ , that is,  $A$  opposite  $a$ ,  $B$  opposite  $b$ , and  $C$  opposite  $c$ .

(1.) Given  $b=118^\circ 2' 14''$ ,  $c=120^\circ 18' 33''$ , and the included angle  $A=27^\circ 22' 34''$ , to find the other parts.



Here we have two sides, and the included angle.

This triangle is represented by  $ABC$ , but we operate on the supplemental triangle  $A'BC$ .

We may let fall the perpendicular  $CD$ , dividing the triangle  $A'BC$  into the two right angled spherical triangles,  $A'DC$ , and  $BDC$ .

Or we may solve either triangle  $ABC$  or  $A'BC$  directly, by applying Equations (8) and (9), (page 350, Geometry), which are

$$\text{Tan. } \frac{1}{2}(C+B) = \frac{\cot. \frac{1}{2}A. \cos. \frac{1}{2}(c-b)}{\cos. \frac{1}{2}(c+b)} \quad (8)$$

$$\text{Tan. } \frac{1}{2}(C-B) = \frac{\cot. \frac{1}{2}A. \sin. \frac{1}{2}(c-b)}{\sin. \frac{1}{2}(c+b)} \quad (9)$$

$$c=120^\circ 18' 33''.$$

$$\text{Whence, } A'B=59^\circ 41' 27''.$$

$$b=118^\circ 2' 14''$$

$$\text{Sum } \quad 238^\circ 20' 47'' \quad \text{half sum } 119^\circ 10' 23.5''.$$

$$\text{Diff. } \quad 2^\circ 16' 19'' \quad \text{half diff. } 1^\circ 8' 9.5''.$$

Cot. $\frac{1}{2}A = 13^\circ 41' 17''$	10.613406		10.613406
cos. $1^\circ 8' 9.5''$	9.999915	sin.	8.297218
	20.613321		18.910624
Cos. $119^\circ 10' 23.5''$	9.687932 <sub>n</sub>	sin.	9.941090
Tan. $\frac{1}{2}(C+B)(83^\circ 13' 42'')$	10.925389 <sub>n</sub>		
Tan. $\frac{1}{2}(C-B)(5^\circ 19' 34'')$			8.969534

NOTE.—The cosine of an arc greater than  $90^\circ$  is negative. Hence the cosine  $119^\circ$  is minus, and we place  $n$  against the log. to show that it is negative. And since  $\tan. \frac{1}{2}(C+B)$  is negative also, the arc must terminate in the 2d quadrant · it is therefore the supplement of  $83^\circ 13' 42''$ ; hence

$$\begin{array}{r} \frac{1}{2}(C+B) \quad 96^\circ 46' 18'' \\ \frac{1}{2}(C-B) \quad 5^\circ 19' 34'' \\ \hline \text{Sum } C, \quad 102^\circ 5' 52'' \\ \text{Diff. } B, \quad 91^\circ 26' 44'' \end{array} \left. \vphantom{\begin{array}{r} \frac{1}{2}(C+B) \\ \frac{1}{2}(C-B) \\ \hline \text{Sum } C \\ \text{Diff. } B \end{array}} \right\} \text{Ans.}$$

For the side  $BC$ , we subtract  $C$  from  $180^\circ$ , giving  $77^\circ 54' 8''$  for the angle  $A'CB$ .

$$\text{Sin. } 77^\circ 54' 8'' : \text{sin. } 59^\circ 41' 27'' :: \text{sin. } 27^\circ 22' 34'' : \text{sin. } \alpha.$$

Sin. $59^\circ 41' 27''$	9.936170
Sin. $27^\circ 22' 34''$	9.662597
	19.598767
Sin. $77^\circ 54' 8''$	9.990246
Sin. $\alpha, 23^\circ 57' 13''$	9.608521. <i>Ans.</i>

(2.) Given

$$\begin{array}{l} A=81^\circ 38' 17'' \\ B=70^\circ 9' 38'' \\ C=64^\circ 46' 32'' \end{array} \left. \vphantom{\begin{array}{l} A \\ B \\ C \end{array}} \right\} \text{to find } \alpha, b, \text{ and } c.$$

By formula (W), (page 348, Geom.)

$$\begin{array}{l} 2)216^\circ 34' 27'' \\ S \quad 108^\circ 17' 13.5'' \\ \quad 81^\circ 38' 17'' \\ \hline S-A \quad 26^\circ 38' 56.5'' \end{array} \quad \text{sin. } \frac{1}{2}\alpha = \left( \frac{-\cos. S. \cos. (S-A)}{\sin. B. \sin. C} \right)^{\frac{1}{2}}.$$

NOTE.—The arc  $S$  being greater than  $90^\circ$ , its cosine is *minus*, and subtracting a *minus* quantity as the sign in the formula indicates, makes it plus.

Cos. $S$ , $108^\circ 17' 13.5''$	9.496623
Cos. $(S-A)$ , $26^\circ 38' 56.5''$	9.951226
	19.447849

Sin. $B$ , $70^\circ 9' 38''$	9.973427	}	sum,	
Sin. $C$ , $64^\circ 46' 32''$	9.956479			
				2) —1.517943

Sin. $\frac{1}{2}a$ . (Radius unity) =	—1.758971
For the radius of tables, add	10.
Tabular sin. of $\frac{1}{2}a = 35^\circ 2' 6.3''$	9.758971

$$\alpha = \frac{2}{70^\circ 4' 13''}. \text{ Ans.}$$

Because,  $\frac{\sin. b}{\sin. B} = \frac{\sin. a}{\sin. A}$

Therefore, Sin.  $b = \sin. B \cdot \frac{\sin. a}{\sin. A}$       Sin.  $c = \sin. C \cdot \frac{\sin. a}{\sin. A}$

Sin. $a$ , $70^\circ 4' 13''$	9.973179		
Sin. $A$ , $81^\circ 38' 17''$	9.995358		
	—1.977821		—1.977821
Sin. $B$	9.973427	sin. $C$	9.956479
$b$ , $63^\circ 21' 24''$	9.951248	$c = 59^\circ 16' 21''$	9.934300
			<i>Ans.</i>

(3.) Given  $\alpha$ ,  $93^\circ 27' 34''$ ,  $b$ ,  $100^\circ 4' 26''$ , and  $c$ ,  $96^\circ 14' 50''$ , the three sides to find the three angles.

By formula, on page 343.

$$\text{Cos. } \frac{1}{2}A = \left( \frac{\sin. S \cdot \sin. (S-a)}{\sin. b \cdot \sin. c} \right)^{\frac{1}{2}} (\text{Radius unity.})$$

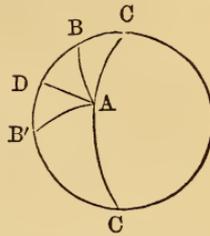
$a$	$93^\circ 27' 34''$	
$b$	$100^\circ 4' 26''$	sin. 9.993252
$c$	$96^\circ 14' 50''$	sin. 9.997413
	<u><math>2)289^\circ 46' 50''</math></u>	<u>19.990665</u> sum,
$S.$	$144^\circ 53' 25''$	sin. 9.759777
	<u><math>93^\circ 27' 34''</math></u>	
$(S-a)$	$51^\circ 25' 51''$	sin. 9.893127
		<u>19.652904</u>
Sub. sum as above,		<u>19.990665</u>
		<u>2)-1.662239</u>
Cos. $\frac{1}{2}A$ ,	$47^\circ 19' 32''$	9.831120 adding 10.
	<u>2</u>	
	<u><math>A, 94^\circ 39' 4''</math></u>	

We may now obtain the other angles by Equations (8) and (9), (page 350, Geom.)

	$b=100^\circ 4' 26''$	
	$c=96^\circ 14' 50''$	
	$\frac{1}{2}(b+c)=98^\circ 9' 38''$	
	$\frac{1}{2}(b-c)=1^\circ 54' 48''$	
Cot. $\frac{1}{2}A$ ,	9.964707	9.964707
Cos. $\frac{1}{2}(b-c)$	9.999758	sin. 8.523587
	<u>19.964465</u>	<u>18.488294</u>
Cos. $\frac{1}{2}(b+c)$	9.152128 $n$	sin. 9.995580
Tan. $\frac{1}{2}(B+C)$	10.812337 $n$	tan. $\frac{1}{2}(B-C)$ 8.492714
	$\frac{1}{2}(B+C)=98^\circ 45' 27''$	
	$\frac{1}{2}(B-C)=1^\circ 46' 52''$	
	<u><math>B=100^\circ 32' 19''</math></u>	
	<u><math>C=96^\circ 58' 35''</math></u>	

(4.) Given two sides,  $b, 84^\circ 16'$ ,  $c, 81^\circ 12'$ , and the angle  $C, 80^\circ 28'$ , to find the other parts.

When this triangle is constructed, we find that the data will correspond equally well to the triangle  $ABC$ , and to  $AB'C$ , in the adjoining cut.



Hence the result is said to be *ambiguous*. In such cases the operator is expected to determine both results.

Observe that  $AB=AB'$ , hence the triangle  $ABB'$  is *isosceles*, and  $AD$  the perpendicular from  $A$  bisects  $BB'$  in  $D$ , making two right angled spherical triangles,  $ADC$  and  $ADB'$ . Their sum is the triangle  $ACB'$ , and their difference  $ABC$ .

For the angles at  $B$ , or at  $B'$ , we have the proportion

$$\text{Sin. } c, 81^\circ 12' : \text{sin. } C, 80^\circ 28' :: \text{sin. } b, 84^\circ 16' : \text{sin. } B.$$

$$\text{Sin. } B = \frac{\text{sin. } 84^\circ 16' \cdot \text{sin. } 80^\circ 28'}{\text{sin. } 81^\circ 12'}$$

$$\text{Sin. } 84^\circ 16' \qquad \qquad \qquad 9.997822$$

$$\text{Sin. } 80^\circ 28' \qquad \qquad \qquad 9.993960$$

$$\hline 19.991782$$

$$\text{Sin. } 81^\circ 12' \qquad \qquad \qquad 9.994857$$

$$\text{Sin. } B', \text{ or } ABD, 83^\circ 11' 24'' \qquad \qquad \qquad 9.996925$$

$$\text{Supplement, } 96^\circ 48' 36'' = ABC. \quad \text{Ans.}$$

By Equation (16), Napier's Circular Parts (page 335, Geometry), we have

$$\text{Tan. } CD \cdot \text{cot. } b = R \cdot \text{cos. } C = R \cdot \text{cos. } 80^\circ 28'$$

$$\text{Tan. } CD = \frac{R \cdot \text{cos. } 80^\circ 28'}{\text{cot. } 84^\circ 16'} \qquad \qquad \qquad \frac{19.219116}{9.001738}$$

$$CD, \text{ tan. } 58^\circ 46' 31'' \qquad \qquad \qquad \hline 10.217378$$

In like manner, we obtain  $BD$ , from the triangle  $ABD$ .

That is,  $\text{Tan. } BD = \frac{R.\cos. 83^\circ 11' 24''}{\cot. c. 81^\circ 12'}$  19.074002  
9.189794

$BD, \tan. 37^\circ 27' 2''$  9.884208

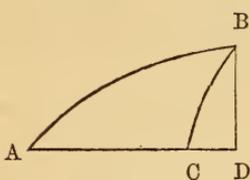
But  $CD, 58^\circ 46' 31''$   
 Sum  $CB',$  or  $a, 96^\circ 13' 33''.$  *Ans.*  
 Diff.  $CB,$  or  $a, 21^\circ 19' 29''.$  *Ans.*

For the angles  $A,$  we have

$$\frac{\text{Sin. } A}{\text{Sin. } a} = \frac{\text{sin. } C}{\text{sin. } c}, \text{ or } \text{sin. } A = \text{sin. } a \cdot \frac{\text{sin. } C}{\text{sin. } c}$$

$\text{Sin. } a. 21^\circ 19' 29''$  9.560688  $\text{sin. } a. 96^\circ 13' 33''$  9.997431  
 $\text{Sin. } C - \text{sin. } c$  -1.999103 -1.999103  
 $A, \text{sin. } 21^\circ 16' 43''$  9.559791  $A, \text{sin. } 97^\circ 13' 45''$  9.996534  
*Ans.*

(5.) Given one side  $c, 64^\circ 26',$  and the adjacent angles  $A, 49^\circ,$  and  $B, 52^\circ,$  to find the other parts.



Let  $ABC$  represent the triangle, and from one extremity of the given side, let fall the perpendicular  $BD,$  making the two right angled spherical triangles,  $ADB$  and  $CDB.$

By Equation (16), Napier's Circular Parts (Geom., p. 335), we have  $\text{Tan. } AD. \cot. 64^\circ 26' = R.\cos. 49^\circ.$

$R.\cos. 49^\circ$  19.816943  
 $\text{Cot. } 64^\circ 26'$  9.679795  
 $\text{Tan. } AD, 53^\circ 54'$  10.137148

For the angle  $ABD,$  we have

$\text{Sin. } 64^\circ 26' : R. :: \text{sin. } 53^\circ 54' : \text{sin. } ABD.$   
 $R.\text{sin. } 53^\circ 54'$  19.907406  
 $\text{Sin. } 64^\circ 26'$  9.955247  
 $\text{Angle } ABD = 63^\circ 35' 51''$  9.952159  
 $ABC \quad 52$   
 $\text{Angle } CBD = 11^\circ 35' 51''$

In the triangle  $ABD$ , we have

$$\begin{array}{r}
 R. : \sin. 64^\circ 26' : : \sin. A, 49^\circ : \sin. BD. \\
 \sin. 49^\circ \qquad \qquad \qquad 9.877780 \\
 \sin. 64^\circ 26' \qquad \qquad \qquad 9.955247 \\
 \hline
 BD, \sin. 42^\circ 54' 26'' \qquad \qquad 9.833027
 \end{array}$$

For  $BC$ , or  $a$  of the triangle  $ABC$ , we use the equation

$$\begin{array}{r}
 \text{Tan. } BD. \cot. BC = R. \cos. 11^\circ 35' 51''. \\
 R. \cos. 11^\circ 35' 51'' \qquad \qquad 19.991042 \\
 \text{Tan. } 42^\circ 54' 26'' \qquad \qquad 9.968246 \\
 \hline
 \text{Cot. } BC, 43^\circ 29' 49'' \qquad \qquad 10.022796. \text{ Ans.}
 \end{array}$$

For  $AC$ , or  $b$ , we have

$$\begin{array}{r}
 \sin. 49^\circ : \sin. 43^\circ 29' 49'' : : \sin. 52^\circ : \sin. AC. \\
 \sin. 43^\circ 29' 49'' \qquad \qquad 9.837788 \\
 \sin. 52^\circ \qquad \qquad \qquad 9.896532 \\
 \hline
 \qquad \qquad \qquad 19.734320 \\
 \sin. 49^\circ \qquad \qquad \qquad 9.877780 \\
 \hline
 b, \text{ or } AC, \sin. 45^\circ 56' 46'' \qquad 9.856540. \text{ Ans.}
 \end{array}$$

For the angle  $C$ , we have

$$\begin{array}{r}
 R. \cos. BC = \cot. CBD. \cot. BCD. \\
 R. \cos. 43^\circ 29' 49'' \qquad \qquad 19.860584 \\
 \text{Cot. } CBD, 11^\circ 35' 51'' \qquad \qquad 10.687769 \\
 \hline
 \text{Cot. } BCD, 81^\circ 31' 56'' \qquad \qquad 9.172815 \\
 \hline
 BCA = 98^\circ 28' 4'' \text{ Ans.}
 \end{array}$$

(6.) Result obvious.

(7.) Given two sides and an angle opposite one of them to determine the other parts.

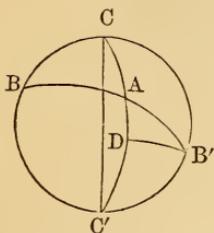
$a = 77^\circ 25' 11''$ ,  $c = 128^\circ 13' 47''$ , and the angle  $C = 131^\circ 11' 12''$ .

To find the angle  $A$ , we have

$$\frac{\sin. A}{\sin. a} = \frac{\sin. C}{\sin. c} = \frac{\sin. 131^\circ 11' 12''}{\sin. 128^\circ 13' 47''} = \frac{\cos. 41^\circ 11' 12''}{\cos. 38^\circ 13' 47''}$$

$$\begin{array}{r} \text{Whence, Sin. } A = \sin. a. \frac{\cos. 41^\circ 11' 12''}{\cos. 38^\circ 13' 47''} \quad \begin{array}{r} 9.989446 \\ 9.876546 \\ \hline 19.865992 \end{array} \quad \text{sin. } a \\ \cos. 38^\circ 13' 47'' \quad \begin{array}{r} 9.895166 \\ \hline 9.970826. \end{array} \quad \text{Ans.} \\ \text{Sin. } A, \quad 69^\circ 13' 59'' \end{array}$$

We have now to determine  $B$  and  $b$ , and the process will be apparent after we construct the triangle, as represented in the adjoining figure.



The data gives the triangle  $ABC$ ;  $AB'C$  is a supplemental triangle. From  $B'$ , let fall the perpendicular  $B'D$ .

In the right angled triangle  $AB'D$ , we have the angle  $A$ , and the hypotenuse  $AB'$ .  $AB'$  being the supplement of  $c$ .

From the two triangles  $ADB'$  and  $C'DB'$ , we can obtain  $AD$ , and  $DC'$ , and their sum taken from  $180^\circ$ , will give  $AC$ , or  $b$ .

The angle  $DC'B$ , is the supplement of  $C$ , which is

$$48^\circ 48' 48''$$

By Napier's Circular Parts, we have

$$\text{Cot. } AB'. \tan. AD = R. \cos. A.$$

$$\begin{array}{r} \text{Or, } \tan. AD = \frac{R. \cos. 69^\circ 13' 59''}{\cot. 51^\circ 46' 13''} \quad \begin{array}{r} 19.549698 \\ 9.896396 \\ \hline 9.653302 \end{array} \\ \tan. AD, 24^\circ 13' 56'' \end{array}$$

$$\begin{array}{r} \text{Also, } \tan. C'D = \frac{R. \cos. 48^\circ 48' 48''}{\cot. a., 77^\circ 25' 11''} \quad \begin{array}{r} 19.818566 \\ 9.348626 \\ \hline 10.469940 \end{array} \\ \tan. C'D, 71^\circ 16' 44'' \end{array}$$

$$AD + DC' = 95^\circ 30' 40''$$

Supplement  $AC'$ ,  $84^\circ 29' 20'' = b$ . *Ans.*

Lastly, for the angle  $B$ , we have

$$\frac{\text{Sin. } B}{\text{Sin. } b} = \frac{\text{sin. } A}{\text{sin. } a}, \text{ or } \text{sin. } B = \frac{\text{sin. } b \cdot \text{sin. } A}{\text{sin. } a}.$$

Sin. $b$	9.997988	
Sin. $A$	9.970826	
	19.968814	
Sin. $a$	9.989446	
Sin. $B$ , $72^\circ 28' 42''$	9.979368.	<i>Ans.</i>

(8.) Given  $\alpha = 68^\circ 34' 13''$  to find  $A$ ,  $B$ , and  $C$ .

$$b = 59^\circ 21' 18'' \quad \text{sin. com.} \quad 065328$$

$$c = 112^\circ 16' 32'' \quad \text{sin. com.} \quad 033684$$

$$2) \overline{240^\circ 12' 3''}$$

Use formula (T),  $120^\circ 6' 1.5''$   $S. \text{ sin.}$  9.937090

(p. 343, Geom.)  $68^\circ 34' 13''$

$$\overline{51^\circ 31' 48.5''} \text{ (} S - \alpha \text{) sin.} \quad 9.893726$$

$$2) \overline{19.929828}$$

$$\text{Cos. } 22^\circ 43' 19'' \quad \overline{9.964914}$$

2

$$A, \overline{45^\circ 26' 38''}. \quad \text{Ans.}$$

Now,  $\text{Sin. } B = \text{sin. } b \cdot \frac{\text{sin. } A}{\text{sin. } a}$ , and  $\text{sin. } C = \text{sin. } c \cdot \frac{\text{sin. } A}{\text{sin. } a}$ .

$$\text{Sin. } b = \quad 9.934672 \quad \text{sin. } c = \quad 9.966316$$

$$\text{Sin. } A - \text{sin. } \alpha = \quad \overline{-1.883936} \quad \overline{-1.883936}$$

$$B, \text{ sin. } 41^\circ 11' 30'' \quad 9.818608 \quad C, 134^\circ 53' 55'' \quad 9.850252$$

*Ans.*

NOTE.—We take  $C$  greater than  $90^\circ$ , because  $c$  was given greater than  $90^\circ$ . The logarithm gives  $45^\circ 6' 5''$ , and its supplement is the angle required.

(9.) Same formula as applied to the preceding.

Given  $\alpha = 89^\circ 21' 37''$  to find  $A$ ,  $B$ , and  $C$ .

$$b = 97^\circ 18' 39'' \quad \text{sin. com.} \quad 003546$$

$$c = 86^\circ 53' 46'' \quad \text{sin. com.} \quad 000637$$

$$\underline{2)273^\circ 34' 2''}$$

$$136^\circ 47' 1'' \text{ S. sin.} \quad 9.835536$$

$$\underline{89^\circ 21' 37''}$$

$$47^\circ 25' 24'' (S-a) \text{ sin.} \quad 9.867097$$

$$\underline{2)19.706816}$$

$$\text{Cos. } \frac{1}{2}A, 44^\circ 28' 40'' \quad 9.853408$$

2

$$A, 88^\circ 57' 20''. \quad \text{Ans.}$$

Now we have,

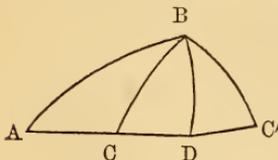
$$\text{Sin. } B = \text{sin. } b \cdot \frac{\text{sin. } A}{\text{sin. } a}, \text{ and } \text{sin. } C = \text{sin. } c \cdot \frac{\text{sin. } A}{\text{sin. } a}.$$

$$\text{Sin. } b \quad 9.996454 \quad \text{sin. } c \quad 9.999362$$

$$\text{Sin. } A - \text{sin. } a \quad -1.999955 \quad -1.999955$$

$$B, \text{ sin. } 97^\circ 21' 26'' \quad 9.996409 \quad C = 88^\circ 47'' 17'' \quad 9.999317$$

(10.) Given  $\alpha = 31^\circ 26' 41''$ ,  $c = 43^\circ 22' 13''$ , and the angle  $A = 12^\circ 16'$ , to find the other parts.



This example applies to the adjacent figure.

$ABC$  and  $ABC'$ , either one, will correspond with so much of the data as is given. Hence the result is ambiguous.

But,

$$\text{Sin. } C = \text{sin. } c \cdot \frac{\text{sin. } A}{\text{sin. } a}.$$

$c$ , sin. $43^{\circ} 22' 13''$	9.836774
$A$ , sin. $12^{\circ} 16'$	9.327281
	<hr/>
	19.164055
$a$ , sin. $31^{\circ} 26' 41''$	9.717400
$C'$ , sin. $16^{\circ} 14' 27''$	9.446655

Or,  $C$ ,  $163^{\circ} 45' 33''$ . *Ans.*

In the right angled spherical triangle  $ADB$ , we have

$$R. : \sin. c :: \sin. A. : \sin. BD.$$

Sum as above, omitting radius,

$$\sin. BD, 8^{\circ} 23' 22'' = 9.164055.$$

In the same triangle we have,

$$R.\cos. c = \cos. AD.\cos. BD.$$

$R.\cos. c$	19.861493
$\cos. BD$	9.995327
	<hr/>
$\cos. AD = 42^{\circ} 42' 37''$	9.866166

In the right angled triangle  $CBD$ , we have

$$R.\cos. a = \cos. CD.\cos. BD.$$

$R.\cos. a$	19.931022
$\cos. BD$	9.995327
	<hr/>
$\cos. CD, 30^{\circ} 24' 57''$	9.935695

$$AD = 42^{\circ} 42' 37''$$

$$\text{Sum} = AC' = b, 73^{\circ} 7' 34''. \quad \text{Ans.}$$

$$\text{Diff.} = AC = b, 12^{\circ} 17' 40''. \quad \text{Ans.}$$

To find the angles at  $B$ , we have the following proportion.

$$\sin. a : \sin. A :: \sin. 12^{\circ} 17' 40'' : \sin. ABC.$$

$\sin. A$	9.327281
$\sin. 12^{\circ} 17' 40''$	9.328248
	<hr/>
	18.655529
$\sin. a$	9.717400
	<hr/>
$ABC, 4^{\circ} 58' 30''$	8.938129

$$\text{Sin. } \alpha : \text{sin. } A. :: \text{sin. } 73^\circ 7' 34'' : \text{sin. } ABC'$$

$$\text{Sin. } A \qquad \qquad \qquad 9.327281$$

$$\text{Sin. } 73^\circ 7' 34'' \qquad \qquad \qquad 9.980888$$

$$\hline 19.308169$$

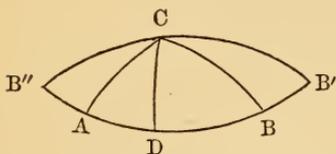
$$\text{Sin. } \alpha \qquad \qquad \qquad 9.717400$$

$$C'BA, \ 22^\circ 56' 16'' \qquad \qquad \qquad 9.590769$$

$$180^\circ$$

$$ABC' \ 157^\circ 3' 44''$$

(11.) In a triangle  $ABC$ , we have given,  $A$ ,  $56^\circ 18' 40''$ ,  $B$ ,  $39^\circ 10' 38''$ , and  $AD$ ,  $32^\circ 54' 16''$ , a segment of the base made by a perpendicular let fall from the angle  $C$ , on to the side  $AB$ , to determine the triangle. The angle  $C$  being obtuse.



This is an ambiguous case. For, in the lune  $B''B$ , let  $B'' = B' = 39^\circ 10' 38''$ . Take a point,  $C$ , nearer to  $B''$  than to  $B'$ , and draw  $CB$  equal to  $CB''$ . Then the triangle  $CB''B$  will be isosceles, and we shall have  $CBB'' = B'' = B'$ . Suppose  $CA$ , and the perpendicular  $CD$ , be drawn, making  $CAB = 56^\circ 18' 40''$ , and  $AD = 32^\circ 54' 16''$ . Now since  $B' = CBA$ , the given parts belong equally to the two triangles,  $CAB$ , and  $CAB'$ . It will be observed, however, that  $CB$  and  $CB'$  are supplements of each other, because  $CB = CB''$ .

$$\text{Tan. } 32^\circ 54' 16''. \text{cot. } AC = R. \text{cos. } A. 56^\circ 18' 40''.$$

$$R.\text{cos. } 56^\circ 18' 40'' \qquad \qquad \qquad 19.744045$$

$$\text{Tan. } 32^\circ 54' 16'' \qquad \qquad \qquad 9.810931$$

$$(b) \text{cot. } AC, 49^\circ 23' 41'' \qquad \qquad \qquad \hline 9.933114. \text{ Ans.}$$

For  $CD$ , we have

$$R. : \sin. AC :: \sin. A : \sin. CD.$$

$$\text{Sin. } AC \qquad \qquad \qquad 9.880363$$

$$\text{Sin. } A \qquad \qquad \qquad 9.920155$$

$$\text{Sin. } CD, 39^\circ 10' 35'' \qquad \qquad \qquad \underline{9.800518}$$

Now in the right angled triangle  $CDB$ , we have

$$\text{Sin. } B : \sin. CD :: R. : \sin. CB.$$

$$R.\text{sin. } CD \qquad \qquad \qquad 19.800518$$

$$\text{Sin. } B \qquad \qquad \qquad 9.800525$$

$$\text{Sin. } CB, 89^\circ 40' \qquad \qquad \qquad \underline{9.999993}$$

Or,  $CB' = 90^\circ 20'.$

NOTE.—It will be observed that arcs which differ from  $90^\circ$  by less than  $1^\circ$ , can not be determined accurately to seconds, when the *sine* is used; and we can not use the cosine or tangent in this case.

For the angle  $ACD$ , we have

$$\text{Sin. } AC : R. :: \sin. AD : \sin. ACD.$$

$$R.\text{sin. } AD, 32^\circ 54' 16'' \qquad \qquad \qquad 19.734991$$

$$\text{Sin. } AC, 49^\circ 23' 41'' \qquad \qquad \qquad 9.880363$$

$$\text{Sin. } ACD, 45^\circ 41' 12'' \qquad \qquad \qquad \underline{9.854628}$$

To determine the angles,  $DCB$  and  $DCB'$ , we have

$$R.\text{cos. } DCB = \tan. CD.\text{cot. } CB.$$

$$R.\text{cos. } DCB' = \tan. CD.\text{cot. } CB'.$$

$$\text{Tan. } CD = 9.911102$$

$$\text{Tan. } CD = 9.911102$$

$$\text{Cot. } CB = 7.764761$$

$$\text{Cot. } CB' = 7.764761n$$

$$\text{Cos. } DCB = 7.675863$$

$$\text{Cos. } DCB' = 7.675863n$$

$$DCB = 89^\circ 44'$$

$$DCB' = 90^\circ 16'$$

Add  $ACD = 45^\circ 41'$

$$ACD = 45^\circ 41'$$

$$ACB = 135^\circ 25' \quad \text{Ans.}$$

$$ACB' = 135^\circ 57' \quad \text{Ans.}$$



Sin. $B$ ,	9.932197	sin. $C$ ,	9.999767
Sin. $a$ —sin. $A$ .	—1.999284		—1.999284
$b=58^\circ 39' 16''$	<u>9.931481</u>	$c=86^\circ 12' 50''$	<u>9.999051</u>

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### SECTION V.

#### APPLICATION OF SPHERICAL TRIGONOMETRY TO THE SOLUTION OF PROBLEMS IN ASTRONOMY.

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(Page 373.)

(2.) In latitude  $42^\circ 40'$  N., when the sun's declination is  $23^\circ 12'$  N., what time will the sun set ?

Tan. $42^\circ 40'$	9.964588
Tan. $23^\circ 12'$	9.632053
Sin. $23^\circ 16' 7''$	<u>9.596641</u>

This arc,  $23^\circ 16' 7''$ , reduced to time at the rate of 4 minutes to one degree, gives  $1^h 33^m 4^s$ .

Add 6

Sun sets,  $7^h 33^m 4^s$  sun rises,  $4^h 26^m 56^s$ .

(3.) What time will the sun set in latitude  $42^\circ 4'$  north, and sun's declination  $15^\circ 21'$  south ?

Tan. $42^\circ 40'$	9.964588
Tan. $15^\circ 21'$	9.438554
Sin. $14^\circ 39' 21''$	<u>9.403142</u>

This arc corresponds to  $58^m 37^s$  in time, which is the interval between six o'clock and sun set, and as the observer is north, and the declination south, the sun must set before six, that is, at  $5^h 1^m 23^s$ , apparent time. The sun must rise same day at  $6^h 58^m 37^s$ , A.M.

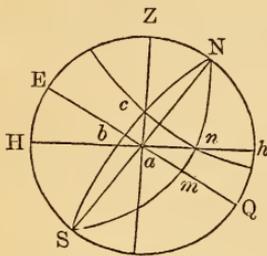
(4.) Lat.  $52^\circ 30'$  north (London), and the sun's declination  $18^\circ 41'$  south. Required, times of sunrise and sunset.

Tan. $52^\circ 30'$	10.115019
Tan. $18^\circ 42'$	<u>9.529535</u>
Sin. $26^\circ 10' 30''$	9.644554

This arc corresponds in time to  $1^h 44^m 42^s$ , to which add  $6^h$  for sunrise, and subtract it from six hours for sunset.

Whence,  $\left. \begin{array}{l} \text{Rises, } \{ 7^h 44^m 42^s \} \\ \text{Sets, } \{ 4^h 15^m 18^s \} \end{array} \right\} \text{apparent time. } \textit{Ans.}$

(5.) This problem is clearly represented by the adjoining cut.  $NS$  is the earth's axis,  $NES$  the meridian,  $EQ$  the equator,  $cn$  the parallel of declination.



The right angled spherical triangle  $amn$ , will give the position of the sun, and times of sunrise and sunset, and the right angled triangle,  $abc$ , or  $ac$ , will be the altitude of the sun, when east and west.

The arc  $am$  corresponds to the times after and before six, when the sun sets and rises, and  $an$  is the arc on the horizon towards the north from the east and west points, when the sun rises and sets.

In short, the solution will be as in the preceding exam-

ples, and the triangle  $amn$  in this cut will illustrate all of them.

Tan. $D$ , $23^\circ 24'$	9.636226
Tan. $L$ , $59^\circ 56'$	10.237394
Sin. $48^\circ 22' 32''$ , $am$ ,	<u>9.873620</u>

In time,  $3^h 13^m 30^s$ .

Adding this interval to 6 hours, gives the time of sunset, and subtracting it from 6 hours, will give the time of sunrise.

For the point  $n$  on the horizon, we have

$$\text{Sin. } man, \text{ or cos. lat. : sin. } D. :: R. : \text{sin. } an.$$

$$\text{Sin. } an = \frac{R \cdot \text{sin. } D}{\text{cos. } L} \quad \begin{array}{r} 19.598952 \\ 9.699844 \end{array}$$

$$N. \text{ of } E., \text{ sin. } 52^\circ 26' 18'' \quad \underline{9.899108}$$

In the right angled spherical triangle  $abc$ , we have the angle  $EaZ$ , the latitude, and  $bc$ , the declination.

$$\text{Sin. } L. : \text{sin. } bc, 23^\circ 24' :: R. : \text{sin. } ac.$$

$ac$  is the altitude of the sun when the sun is at the point  $c$ , east or west on the prime vertical.

$$\text{Sin. Alt.} = \frac{R \cdot \text{sin. } D}{\text{sin. } L} \quad \begin{array}{r} 19.598952 \\ 9.937238 \end{array}$$

$$\text{Sin. } ac, 27^\circ 18' 57'' \quad \underline{9.661714}$$

To find  $ba$ , the time before and after six o'clock, apparent time, when the sun is east and west, we have

$$R. : \text{cos. } ab :: \text{cos. } bc : \text{cos. } ac.$$

$$\text{Cos. } ab = \frac{R \cdot \text{cos. } ac}{\text{cos. } bc} \quad \begin{array}{r} 19.948653 \\ 9.962727 \end{array}$$

$$\text{Cos. } ab, 14^\circ 30' 30'' \quad \underline{9.985926}$$

This arc ( $14^\circ 30' 33''$ ) reduced to time, is equivalent to 58 minutes 2 seconds. Hence the sun is east at  $6^h 58^m 2^s$ , A.M., and it is west at  $5^h 1^m 58^s$ , P.M., in latitude  $59^\circ 56'$

when the sun's declination is  $23^{\circ} 24'$  north. Problem (2), [page 376].

By the formula on same page (Text-book), we must operate as follows,

True Alt.,	$36^{\circ} 12'$		
Lat. north,	$40^{\circ} 21'$	cos. com.	0.117986
North polar dis.	$93^{\circ} 20'$	sin. com.	0.000735
	<u><math>2)169^{\circ} 53'</math></u>		
	$84^{\circ} 56' 30''$	cos.	8.945320
	<u><math>36^{\circ} 12'</math></u>		
	$48^{\circ} 44' 30''$	sin.	9.876070
			<u><math>2)18.940111</math></u>
	Sin. $\frac{1}{2}P$ , $17^{\circ} 10' 1''$		9.470055

The double of this angle, or ( $34^{\circ} 20' 2''$ ), changed into time, is  $2^h 17^m 20^s$ , the interval of time from apparent noon.

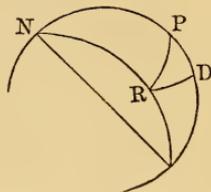
Whence, from noon,	$12^h 0^m 0^s$	
Take,	<u><math>2^h 17^m 20^s</math></u>	
	$9^h 42^m 40^s$	A.M., apparent time.

(3.)	Alt.,	$40^{\circ} 8'$		
	Latitude,	$21^{\circ} 2'$	cos. com.	0.029945
	South polar dis.	$108^{\circ} 32'$	sin. com.	0.013228
		<u><math>2)169^{\circ} 42'</math></u>		
		$84^{\circ} 51'$	cos.	8.953100
		<u><math>40^{\circ} 8'</math></u>		
		$44^{\circ} 43'$	sin.	9.847327
				<u><math>2)18.853500</math></u>
	$\frac{1}{2}P$ , sin.	$15^{\circ} 29' 40''$		9.426750
		$2$		
		<u><math>30^{\circ} 59' 20''</math></u>	$= 2^h 3^m 57^s$ , P.M.	<i>Ans.</i>

A Geographical Problem, (page 377, Text-book.)

Required, the number of degrees on a great circle between New Orleans and Rome, also the number of miles taking 69.16 miles to each degree.

Let  $P$ , be the north pole on the earth,  $N$ , the position of New Orleans, and  $R$  the position of Rome, which positions are



	<i>Latitude.</i>	<i>Longitude.</i>
New Orleans,	$29^{\circ} 57' 30'' N.$	$90^{\circ} W.$
Rome,	$41^{\circ} 53' 54'' N.$	$12^{\circ} 28' 40'' E.$

The co. latitude of N. O. is  $NP=60^{\circ} 2' 30''$ .

co. latitude of Rome is  $PR=48^{\circ} 6' 6''$ .

And the sum of longitudes is  $102^{\circ} 28' 40''$ , which is the angle  $NPR$ . The supplement of the angle  $NPR$ , is  $RPD$ ,  $77^{\circ} 31' 20''$ .  $RD$ , is the perpendicular let fall on  $NP$  produced.

Now, in the triangle  $PRD$ , we have

$$R. : \sin. PR, 48^{\circ} 6' 6'' :: \sin. 77^{\circ} 31' 20'' : \sin. RD.$$

$$\sin. PR \qquad \qquad \qquad 9.871767$$

$$\sin. P \qquad \qquad \qquad 9.989619$$

$$\sin. RD, 46^{\circ} 36' 53'' \qquad \qquad \qquad 9.861386$$

$$R.\cos. PR = \cos. PD.\cos. RD.$$

$$\cos. PD = \frac{R.\cos. PR}{\cos. RD} \qquad \qquad \qquad 19.824654$$

$$\qquad \qquad \qquad 9.836894$$

$$\cos. PD = \cos. 13^{\circ} 32' 21'' \qquad \qquad \qquad 9.987760$$

$$\text{Add } NP, \qquad \qquad \qquad 60^{\circ} 2' 30''$$

$$\text{Sum, or } ND, \qquad \qquad \qquad 73^{\circ} 34' 51''$$

In the right angled triangle  $NDR$ , we have

$$R.\cos. NR = \cos. ND.\cos. DR.$$

Cos. $ND$ , $73^\circ 34' 51''$	9.451268
Cos. $DR$ , $46^\circ 36' 53''$	9.836894
Cos. $NR$ , $78^\circ 48' 15''$	<u>9.288162</u>

That is, the distance in degrees is  $78^\circ.8041$ , and at the rate of 69.16 miles to each degree, gives 5450.1 English miles.

We can compute this side more concisely by using the formula

$$\cos. c = \cos. (a+b) + 2\cos.^2 \frac{1}{2}C.\sin. a.\sin. b.$$

Let the angle  $NPR = C = 102^\circ 28' 40''$ ,  $a = 60^\circ 2' 30''$ , and  $b = 48^\circ 6' 6''$ ,  $c =$  the arc  $NR$ .

Nat. Cos. $(a+b) = \text{Nat. cos. } 108^\circ 8' 36'' = -.31139.$		
Log. 2		0.301030
Cos. $\frac{1}{2}C$ , $R=1$		-1.796627
		<u>-1.796627</u>
Sin. $a$		-1.937712
Sin. $b$		-1.871767
Nat. number,	0.50555	-1.703763. Sum.
Add,	<u>-0.31139</u>	
Nat. cos. $78^\circ 48' 15''$	.19416.	<i>Ans.</i> same as before.

**NOTE.**—There are nine examples on page 379, Text-book; we shall show the solutions of only two or three of them, by a special formula, which we think the most concise, all things considered. Others may solve them by this or any one of the other methods, explained in the Geometry.

The difference of the right ascensions of the two bodies at the time designated, changed into arc, will be the included angle of a spherical triangle, and the complements of the declinations are the two sides of such a triangle; the third side is required and we designate it by  $x$ . The angle at the pole of the celestial equator, we designate by  $P$ , and  $a$  and  $b$  the sides.

Now, by the Fundamental Equation in Spherical Trigonometry, we have. (See page 342, Geom.)

$$\text{Cos. } P = \frac{\cos. x - \cos. a \cos. b}{\sin. a \sin. b}. \quad (1)$$

Let  $D$  be the greater declination, and  $d$  the less.

Then  $D$ , and  $d$ , are complements of  $a$  and  $b$ , and  $\cos. a = \sin. D$ , and  $\sin. a = \cos. D$ , and the above equation becomes,

$$\text{Cos. } P = \frac{\cos. x - \sin. D \sin. d}{\cos. D \cos. d}.$$

Subtracting each member from unity, we obtain

$$1 - \cos. P = 1 - \frac{\cos. x - \sin. D \sin. d}{\cos. D \cos. d}.$$

$$1 - \cos. P = \frac{(\cos. D \cos. d + \sin. D \sin. d) - \cos. x}{\cos. D \cos. d}.$$

That is, 
$$2 \sin.^2 \frac{1}{2} P = \frac{\cos. (D-d) - \cos. x}{\cos. D \cos. d}.$$

Whence,  $\text{Cos. } x = \cos. (D-d) - 2 \sin.^2 \frac{1}{2} P \cos. D \cos. d$  ( $F$ )

This formula is essentially the same as that in the Text-book.

In that, the given sides of the triangle are used. In this, it is the complements of those sides.

#### EXAMPLES.

June 24, 1860. At noon, mean time.

	<i>R. A.</i>		<i>Declination.</i>
Moon,	10 <sup>h</sup> 51 <sup>m</sup> 36.5 <sup>s</sup>	$d =$	3° 35' 24" N.
Jupiter,	8 <sup>h</sup> 4 <sup>m</sup> 27.6 <sup>s</sup>	$D =$	20° 51' 36.8" N.
Diff.	<u>2<sup>h</sup> 47<sup>m</sup> 8.9<sup>s</sup></u>	$(D-d) =$	<u>17° 16' 12.8"</u>
Or,	41° 47' 13.5" = $P$		
	20° 53' 36.8" = $\frac{1}{2}P$ .		

We now apply the formula ( $F$ ).

Log. 2	0.301030
Log. sin. $\frac{1}{2}P$ , $20^\circ 53' 36.8''$	-1.552221
Log. sin. $\frac{1}{2}P$ , $20^\circ 53' 36.8''$	-1.552221
Log. cos. $D$ , $20^\circ 51' 36.8''$	-1.970557
Log. cos. $d$ , $3^\circ 35' 24''$	-1.999147
Nat. num.	<u>0.237233</u>
	-1.375176
Nat. cos. $(D-d)=$	0.954915
Diff.=Nat. cos. $x=$	<u>0.717682=44^\circ 8' 12''</u> . <i>Ans.</i>

NOTES.—1. The three remaining problems in the same group are solved in the same manner.

2. Let the student observe that if the natural sine or cosine is required to more than 5 decimal places, the *logarithmic* sine or cosine should first be taken out; and from this, diminishing the index by 10, the number may be obtained correctly to 6 or 7 places. Conversely, if the *arc* is required from the natural sine or cosine, first find the *logarithm*, then the arc.

We shall now solve one or two in the next group, where the distances are greater than  $90^\circ$ , and the declinations on opposite sides of the equator. The formula employed is the same. Recollect, however, that the cosine of an arc, greater than  $90^\circ$ , must be taken with a *minus sign*.

Example 1st., October 6, 1860. At noon.

	<i>R. A.</i>	<i>Declination.</i>
Sun	$12^h 49^m 29.3^s$	- $d$ $5^\circ 18' 42.6'' S.$
Moon,	$5^h 41^m 20.8^s$	$D$ $26^\circ 8' 00'' N.$
	<u><math>7^h 8^m 8.5^s</math></u>	$(D-d)=$ <u><math>31^\circ 26' 42.6''</math></u>
In arc,	$107^\circ 2' 7.5''=P$	
	$53^\circ 31' 3.8''=\frac{1}{2}P.$	

Log. 2	0.301030
Log. sin. $53^\circ 31' 3.8''$	-1.905278
Log. sin. $53^\circ 31' 3.8''$	-1.905278
Cos. $26^\circ 8'$	-1.953166
Cos. $5^\circ 18' 42.6''$	-1.998130
Nat. $N.$	<u>-1.155798</u>
	-0.062882
Nat. cos. $(D-d)$	0.853139
Diff., cos. $x,$	<u>-0.302659=107^\circ 37' 2''</u> .

We will work the next example in this group by Equations (8) and (9), page 350, Text-book.

Making  $C$  the included angle, and  $A$ , the angle opposite the greater polar distance.

$$\begin{array}{rcl} \odot RA, & 12^h 49^m 56.7^s & \text{Dec. } 5^\circ 21' 35.4'' S. \\ \oplus RA, & 5^h 48^m 30.1^s & \text{Dec. } 26^\circ 3' 20'' N. \\ \hline & 7^h 1^m 26.6^s & \end{array}$$

$$\begin{array}{l} \text{In degrees,} \\ 105^\circ 21' 39'' = C \\ 52^\circ 40' 49.5'' = \frac{1}{2}C. \end{array}$$

The distance from the north pole to the center of the sun, is the sun's declination added to  $90^\circ$ , which we designate by  $a$ .

$$\text{Therefore,} \quad a = 95^\circ 21' 35.4''$$

$$\text{The moon from the same point is } b = 63^\circ 56' 40''$$

$$\text{Half sum, is} \quad \frac{1}{2}(a+b) = 79^\circ 39' 7.7''$$

$$\text{Half diff., is} \quad \frac{1}{2}(a-b) = 15^\circ 42' 27.7''$$

$$\text{Cot. } \frac{1}{2}C, \quad 9.882147 \quad \text{cot. } \frac{1}{2}C, \quad 9.882147$$

$$\text{Cos. } \frac{1}{2}(a-b) \quad 9.983467 \quad \text{sin. } \frac{1}{2}(a-b) \quad 9.432536$$

$$\hline 19.865614 \quad \hline 19.314683$$

$$\text{Cos. } \frac{1}{2}(a+b) \quad 9.254364 \quad \text{sin. } \frac{1}{2}(a+b) \quad 9.992878$$

$$\text{Tan. } \frac{1}{2}(A+B), \quad 10.611250 \quad \text{tan. } \frac{1}{2}(A-B) \quad 9.321805$$

$$\frac{1}{2}(A+B), \quad 76^\circ 14' 47'' \quad \frac{1}{2}(A-B) \quad 11^\circ 50' 56''$$

$$\frac{1}{2}(A-B), \quad 11^\circ 50' 56''$$

$$\text{Diff.} = B = 64^\circ 23' 51''$$

Lastly, for the side sought, we have the proportion

$$\text{Sin. } B, 64^\circ 23' 51'' : \text{sin. } b, 63^\circ 56' 40'' :: \text{sin. } 105^\circ 21' 39'' : \text{sin. } x.$$

$$\text{Sin. } b \quad 9.953454$$

$$\text{Sin. } C = \cos. 15^\circ 21' 39'' \quad 9.984201$$

$$\hline 19.937655$$

$$\text{Sin. } B \quad 9.955116$$

$$\text{Sin. } 106^\circ 8' 19'' = \cos. 16^\circ 8' 19'' \quad 9.982539$$

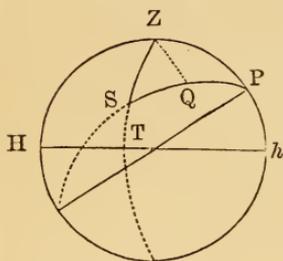
We prefer the other formula.

## ASTRONOMICAL PROBLEMS.

The following Astronomical problems were included in Robinson's Geometry, as first published, but being deemed too difficult for such a work, were omitted in the New Geometry.

(1.) In latitude  $40^{\circ} 48'$  north, the sun bore south  $78^{\circ} 16'$  west, at  $3^h 38^m$  P.M., apparent time. Required his altitude and declination, making no allowance for refraction.

*Ans.* The altitude,  $36^{\circ} 46'$ , and declination,  $15^{\circ} 32'$  north.



Let  $Hh$  be the horizon,  $Z$  the zenith of the observer,  $P$  the north pole, and  $PS$  a meridian through the sun.

$PZ$  is the co-latitude,  $49^{\circ} 12'$ , and  $PS$  is the co-declination or polar distance, one of the arcs sought.

$ZS$  is the co-altitude, or  $ST$  is the altitude of the sun at the time of observation.

The angle  $ZPS$  is found by reducing  $3^h 38^m$  to degrees at the rate of  $4^m$  to one degree; hence,  $ZPS = 54^{\circ} 30'$ .

Because  $HZS = 78^{\circ} 16'$ ,  $PZS = 101^{\circ} 44'$ . From  $Z$  let fall the perpendicular  $ZQ$  on  $PS$ . Then in the right angled spherical  $\triangle PZQ$ , equation (13) gives us\*

$$\begin{array}{r}
 R.\sin. ZQ = \sin. PZ \sin. P \\
 \sin. PZ = \sin. 49^{\circ} 12' \qquad 9.879093 \\
 \sin. P = \sin. 54^{\circ} 30' \qquad 9.910686 \\
 \sin. ZQ = \sin. 38^{\circ} 2' 42'' \qquad 9.789779
 \end{array}$$

\* To apply the equations without confusion, letter each right angled spherical triangle  $ABC$ , right angled at  $B$ , then  $A$  must be written in place of  $P$ ; and when operating on  $ZSQ$ , write  $A$  in place of  $S$ , and  $C$  for the angle  $SZQ$ .

To obtain the angle  $PZQ$ , we apply equation (19), which gives

$$R.\cos. PZQ = \cot. PZ.\tan. ZQ.$$

That is,  $R.\cos. PZQ = \tan. 40^\circ 48' . \tan. 38^\circ 2' 42'' .$

Tan. $40^\circ 48'$	9.936100
---------------------	----------

Tan. $38^\circ 2' 42''$	9.893513
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$PZQ = \cos. 47^\circ 30' 30''$	9.829613
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$PZS = 101^\circ 44'$	
-----------------------	--

$SZQ = 54^\circ 13' 30''$	
---------------------------	--

To obtain  $ZS$  or its complement, we again apply (19)

$$(19) \quad R.\cos. SZQ = \cot. ZS.\tan. ZQ.$$

That is,  $R.\cos. 54^\circ 13' 30'' = \tan. ST.\tan. 38^\circ 2' 42'' .$

$R.\cos. 54^\circ 13' 30'' =$	19.766761
-------------------------------	-----------

tan. $38^\circ 2' 42'' =$	9.893513
---------------------------	----------

Tan. $36^\circ 46'$ , nearly	9.873248
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To find  $PS$ , we take the following proportion,

$$\text{Sin. } P : \text{sin. } ZS :: \text{sin. } PZS : \text{sin. } PS.$$

That is,

$$\text{Sin. } 54^\circ 30' : \cos. 36^\circ 46' :: \text{sin. } 101^\circ 44' : \text{sin. } PS.$$

Cos. $11^\circ 44'$	9.990829
---------------------	----------

Cos. $36^\circ 46'$	9.903676
---------------------	----------

	19.894505
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Sin. $54^\circ 30'$	9.910686
---------------------	----------

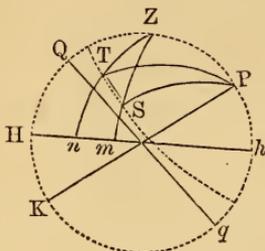
$PS, \text{sin. } 74^\circ 28'$	9.983819
---------------------------------	----------

Whence, the sun's distance from the equator must have been  $15^\circ 32'$  north.

(2.) In north latitude, when the sun's declination was  $14^\circ 20'$  north, his altitudes, at two different times on the

same forenoon, were  $43^{\circ} 7' +$ , and  $67^{\circ} 10' +$  : and the change of azimuth, in the interval,  $45^{\circ} 2'$ . Required the latitude.

*Ans.*  $34^{\circ} 21' 14''$  north.



Let  $PK$  be the earth's axis,  $Qq$  the equator, and  $Hh$  the horizon.

Also, let  $Z$  be the zenith of the observer,  $Sm$  the first altitude,  $Tn$  the second, and the angle  $TZS = 45^{\circ} 2'$ .

Our first operation must be on the triangle  $ZTS$ .  $ZT = 22^{\circ} 50'$ ,  $ZS = 46^{\circ} 53'$ , and we must find  $TS$ , and the angle  $TSZ$ .

From  $T$ , conceive  $TB$  let fall on  $ZS$ , making two right angled  $\triangle$ 's; and to avoid confusion in the figure, we will keep the arc  $TB$  in mind, and not actually draw it.

Then the  $\triangle ZTB$  furnishes this proportion,

$$R : \sin. 22^{\circ} 50' :: \sin. 45^{\circ} 2' : \sin. TB = \sin. 15^{\circ} 56' 8''.$$

To find  $ZB$  we have the following proportion,

$$R : \cos. ZB :: \cos. 15^{\circ} 56' 8'' : \cos. 22^{\circ} 50'.$$

Whence, we find  $ZB = 16^{\circ} 34' 13''$ . Now in the right angled spherical  $\triangle TBS$ , we have  $TB = 15^{\circ} 56' 8''$ ,  $BS = 46^{\circ} 53' - 16^{\circ} 34' 13''$ , or  $BS = 30^{\circ} 18' 47''$ ; and  $TS$  is found from the following proportion,

$$R : \cos. 15^{\circ} 56' 8'' :: \cos. 30^{\circ} 18' 47'' : \cos. TS.$$

This gives  $TS = 33^{\circ} 53' 26''$ . To find the angle  $TSZ$ , we have the proportion,

$$\sin. 33^{\circ} 53' 26'' : R :: \sin. TB 15^{\circ} 56' 8'' : \sin. TSZ.$$

Whence, the angle  $TSZ = 29^{\circ} 29' 49''$ .

The next step is to operate on the *isosceles* spherical  $\triangle PTS$ . We require the angle  $TSP$ .

Conceive a meridian drawn bisecting the angle at  $P$ , it will

also bisect the base  $TS$ , forming two equal right angled spherical triangles.

Observe that  $PS=75^\circ 40'$  and  $\frac{1}{2} TS=16^\circ 56' 43''$ .

To find the angle  $TSP$ , we apply equation (19), in which  $a=16^\circ 56' 43''$ ,  $b=75^\circ 40'$ , and the equation becomes,

$$R.\cos. TSP = \cot. 75^\circ 40'. \tan. 16^\circ 56' 43''.$$

Whence,  $TSP=85^\circ 32' 5''$ , and  $PSZ=85^\circ 32' 5'' - 29^\circ 29' 49'' = 56^\circ 2' 16''$ .

The third step is to operate on the  $\triangle ZSP$ ; we now have its two sides  $ZS$  and  $SP$ , and the included angle.

From  $Z$  conceive a perpendicular arc let fall on  $SP$ , calling it  $ZB$ ; then the right angled spherical triangle  $SZB$ , gives

$$R. : \sin. ZS :: \sin. ZSB : \sin. ZB.$$

That is,

$$R. : \sin. 46^\circ 53' :: \sin. 56^\circ 2' 16'' : \sin. ZB = \sin. 37^\circ 15' 37''.$$

To find  $SB$ , we have the following proportion,

$$R. : \cos. SB :: \cos. ZB : \cos. ZS.$$

That is,  $R. : \cos. SB :: \cos. 37^\circ 15' 37'' : \cos. 46^\circ 53'$ .

Whence,  $SB=30^\circ 49' 18''$ . Now, from  $PS$ ,  $75^\circ 40'$ , take  $SB$ ,  $30^\circ 49' 18''$ , and the difference must be  $BP$ ,  $44^\circ 50' 42''$ .

Lastly, to obtain  $PZ$ , and consequently  $ZQ$  the latitude, we have

$$R. : \cos. ZB :: \cos. BP : \cos. ZP = \sin. ZQ.$$

That is,  $R. : \cos. 37^\circ 15' 37'' :: \cos. 44^\circ 50' 42'' : \sin. ZQ = \sin. 34^\circ 21' 14''$  north.

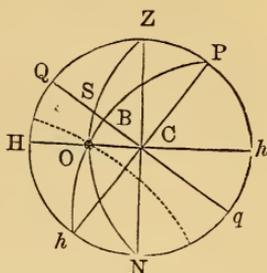
This is the result by a careful computation, and it differs  $1' 14''$  from the answer given in the text-book.

This is a modification of latitude by double altitudes, but in real double altitudes the arc  $TS$  is measured from the elapsed time between the observations, and the angle  $TZS$  is not given.



(4.) The sun set south-west  $\frac{1}{2}$  south, when his declination was  $16^\circ 4'$  south. Required the latitude. *Ans.*  $69^\circ 1'$  north.

Draw a circle as before. Let  $Hh$  be the horizon,  $Z$  the zenith,  $P$  the pole. The great circle  $PZH$  is the meridian, and  $ZCN$  at right angles to it, and of course east and west. Let  $BC$  be a portion of the equator, and  $BO$  the arc of declination. The position on the horizon where the sun set is the arc  $HO = 45^\circ - 5^\circ 37' 30'' = 39^\circ 22' 30''$ .



Consequently, the arc  $OC = 50^\circ 37' 30''$ .

In the right angled spherical triangle  $BOC$ , we have  $BC$ ,  $BO$  given to find the angle  $BCO$ , which is the complement of the latitude, or the complement of the angle  $BCZ$ .

To find the angle  $BCO$ , we apply equation (14).

$$R.\sin. BO = \sin. OC.\sin. BCO.$$

That is,  $R.\sin. 16^\circ 4' = \sin. 50^\circ 37' 30''.\sin. BCO.$

$R.\sin. 16^\circ 4'$	19.442096
$\sin. 50^\circ 37' 30''$	9.888186
$\cos. 69^\circ 1'$ nearly	9.553910

SCHOLIUM.—The arc  $BC$  on the equator measures the angle  $BPC$ , corresponding to the time from 6 o'clock to sunrise or sunset. This arc is called the arc of ascensional difference in astronomy. The time of sunset is before six, if the latitude is north and the declination south, as in this example, but after six, if the latitude and declination are both north or both south.

To obtain this arc, the latitude and declination must be given; that is,  $BO$  and the angle  $BCO$ , the complement of the latitude. Here we apply (12), that is,

$$R.\sin. BC = \tan. D.\tan. L,$$

an equation in which  $D$  represents the declination, and  $L$  the latitude.

(5.) The altitude of the sun, when on the equator, was  $14^{\circ} 28' +$  bearing east  $22^{\circ} 30'$  south. Required the latitude and time. *Ans.* Latitude  $56^{\circ} 1'$ , and time  $7^h 46^m 11^s$ , A.M.

Let  $S$  be the position of the sun on the equator. (See the last figure.) Draw the arc  $ZS$ , and the right angled spherical  $\triangle ZQS$  is the one we have to operate upon.

Then  $ZS$  is the complement of the given altitude, and the angle  $QZS$ , is the complement of  $22^{\circ} 30'$ . The portion of the equator between  $Q$  and  $S$ , changed into time, will be the required time from noon, and the arc  $QZ$  will be the required latitude.

First for the arc  $QS$ .

$$R : \sin. ZS :: \sin. QZS : \sin. QS.$$

That is,

$$R. : \cos. 14^{\circ} 28' :: \cos. 22^{\circ} 30' : \sin. QS = 63^{\circ} 27' 19''.$$

But  $63^{\circ} 27' 19''$  at the rate of  $4^m$  to one degree, corresponds to  $4^h 13^m 49^s$  from noon—and as the altitude was marked  $+$ , rising, it was before noon, or at  $7^h 46^m 11^s$  in the morning.

To find the arc  $QZ$ , we have the following proportion,

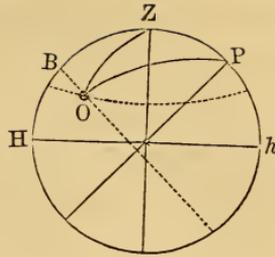
$$R. : \cos. 63^{\circ} 27' 19'' :: \cos. QZ : \sin. 14^{\circ} 28'.$$

Whence,  $\cos. QZ = \cos. 56^{\circ} 1'$  nearly, and  $56^{\circ} 1'$  is the latitude sought.

(6.) The altitude of the sun was  $20^{\circ} 41'$  at  $2^h 20^m$  P.M. when his declination was  $10^{\circ} 28'$  south. Required his azimuth and the latitude.

*Ans.* Azimuth south  $37^{\circ} 5'$  west, latitude  $51^{\circ} 58'$  north.

This problem furnishes the spherical  $\triangle PZO$ , in which the side  $ZO$  is the complement of  $20^\circ 41'$  or  $69^\circ 19'$ .  $PO = 90^\circ + 10^\circ 28' = 100^\circ 28'$ , and the angle  $ZPO$  is  $2^h 20^m$ , changed into degrees at the rate of  $15^\circ$  to one hour, or  $ZPO = 35^\circ$ .



Now in the triangle  $ZPO$ , we have

$$\text{Sin. } ZO : \text{sin. } ZPO :: \text{sin. } PO : \text{sin. } PZO.$$

That is,

$$\text{Cos. } 20^\circ 41' : \text{sin. } 35^\circ :: \text{cos. } 10^\circ 28' : \text{sin. } PZO = \text{sin. } 37^\circ 5'.$$

In the right angled spherical  $\triangle BOZ$ , we apply equation (16).

$$(16). \quad R.\text{cos. } 37^\circ 5' = \text{tan. } 20^\circ 41' \cdot \text{tan. } BZ.$$

$R.\text{cos. } 37^\circ 5'$	19.901872
$\text{Tan. } 20^\circ 41'$	9.576958
$\text{Tan. } BZ = \text{tan. } 64^\circ 40'$	<u>10.324914</u>

To find  $PB$  in the right angled  $\triangle BPO$ , we apply the same equation (16).

$$R.\text{cos. } 35^\circ = \text{tan. } 10^\circ 28' \cdot \text{tan. } PB.$$

$R.\text{cos. } 35^\circ$	19.913365
$\text{Tan. } 10^\circ 28'$	9.266555
$\text{Cot. } 12^\circ 42'$	<u>10.646810</u>

But  $PB$  is obviously greater than  $90^\circ$ , therefore the point  $B$  is  $12^\circ 42'$  below the equator; but from  $B$  to  $Z$  is  $64^\circ 40'$ ; therefore, from  $Z$  to the equator, or the latitude, is the difference between  $64^\circ 40'$  and  $12^\circ 42'$ , or  $51^\circ 58'$  north.

*Ans.* Lat.  $51^\circ 58'$  north.

(7.) If in August 1840, Spica was observed to set  $2^h 26^m$   $14^s$  before Arcturus, what was the latitude of the observer,

no account being taken of the height of the eye above the sea, nor of the effect of refraction? *Ans.*  $36^{\circ} 47' 38''$  north.

By a catalogue of the stars to be found in the author's Astronomy, or in any copy of the English Nautical Almanac, we find the positions of these stars in 1840, to have been as follows :

Spica, right ascension,  $13^h 16^m 46^s$  Dec.  $10^{\circ} 19' 40''$  south.  
 Arcturus, " "  $14^h 8^m 25^s$  Dec.  $20^{\circ} 1' 4''$  north.

Let  $L$  = the latitude sought. Put  $d = 10^{\circ} 19' 40''$ , and  $D = 20^{\circ} 1' 4''$ .

The difference in right ascensions is  $51^m 39^s$ , and this would be about the time that Arcturus would set after Spica, provided the observer was near the equator or a little south of it; but as the interval observed was  $2^h 26^m 14^s$ , the observer must have been a considerable distance in *north latitude*. In high southern latitudes Arcturus sets before Spica.

When an observer is north of the equator, and the sun or star south of it, the sun or star will set within six hours after it comes to the meridian.

When the observer and the object are both north of the equator, the interval from the meridian to the horizon is greater than six hours.

The difference between this interval and six hours, is called the ascensional difference, and it is measured in *arc* by  $BC$  in the figure to the 4th example.

Now let  $x$  = the ascensional difference of Spica corresponding to the latitude  $L$ , and  $y$  = that of Arcturus corresponding to the same latitude; then by the scholium to the 4th example, calling radius unity, we shall have

$$\sin. x = \tan. L. \tan. d \quad (1)$$

$$\sin. y = \tan. L. \tan. D \quad (2)$$

The star Spica came to the observer's meridian at a certain time, that we may denote by  $M$ .

Then 
$$M + \left(6 - \frac{x}{15}\right) = \text{the time Spica set.}$$

And 
$$M + 51^m 39^s + \left(6 + \frac{y}{15}\right) = \text{the time Arcturus set.}$$

By subtracting the time Spica set from the time Arcturus set, we shall obtain an expression equal to  $2^h 26^m 14^s$ .

That is, 
$$51^m 39^s + \frac{x}{15} + \frac{y}{15} = 2^h 26^m 14^s.$$

Or, 
$$\frac{x}{15} + \frac{y}{15} = 1^h 34^m 35^s \tag{3}$$

$$x + y = 15 (1^h 34^m 35^s) \tag{4}$$

Equation (3) expresses time. Equation (4) expresses arc.

When we divide arc by 15, we obtain time, one degree being the *unit* for arc, and one hour the *unit* for time ; therefore, when we multiply time by 15, we obtain arc ; that is,  $1^h$  multiplied by 15, gives  $15^\circ$  ; hence (4) becomes

$$\begin{aligned} x + y &= 23^\circ 38' 45'' = a \\ x &= a - y \end{aligned} \tag{5}$$

That is, the *arc*  $x$  is equal to the difference of the *arcs*  $a$  and  $y$  ; but to make use of these *arcs* and avail ourselves of equations (1) and (2), we must take the *sines* of the arcs, (see equation (8), plane trigonometry) ; then (5) becomes

$$\text{Sin. } x = \text{sin. } a \cdot \text{cos. } y - \text{cos. } a \cdot \text{sin. } y. \tag{6}$$

Substituting the values of  $\text{sin. } x$  and  $\text{sin. } y$  from (1) and (2), (6) becomes

$$\text{Tan. } L \cdot \text{tan. } d = \text{sin. } a \cdot \text{cos. } y - \text{cos. } a \cdot \text{tan. } L \cdot \text{tan. } D \tag{7}$$

$$\text{Squaring (2),} \quad \text{sin.}^2 y = \text{tan.}^2 L \cdot \text{tan.}^2 D.$$

Subtracting each member from unity, and observing that  $(1 - \text{sin.}^2 y)$  equals  $\text{cos.}^2 y$ , then

$$\text{Cos.}^2 y = 1 - \text{tan.}^2 L \cdot \text{tan.}^2 D.$$

Or, 
$$\text{Cos. } y = \sqrt{1 - \text{tan.}^2 L \cdot \text{tan.}^2 D}.$$

This value of  $\cos. y$  put in (7), gives

$$\text{Tan. } L. \tan. d = \sin. a. \sqrt{1 - \tan.^2 L. \tan.^2 D} - \cos. a. \tan. L. \tan. D \quad (8)$$

By transposition and division,

$$\left( \frac{\text{Tan. } d + \cos. a. \tan. D}{\sin. a} \right) \tan. L = \sqrt{1 - \tan.^2 L. \tan.^2 D}$$

Squaring,

$$\left( \frac{\text{Tan. } d + \cos. a. \tan. D}{\sin. a} \right)^2 \tan.^2 L = 1 - \tan.^2 L. \tan.^2 D$$

Dividing by  $\tan.^2 L$ , and observing that  $\frac{1}{\tan.^2 L} = \cot.^2 L$  we

have 
$$\left( \frac{\text{Tan. } d + \cos. a. \tan. D}{\sin. a} \right)^2 = \cot.^2 L - \tan.^2 D$$

Or, 
$$\begin{aligned} \cot.^2 L &= \tan.^2 D + \left( \frac{\text{tan. } d + \cos. a. \tan. D}{\sin. a} \right)^2 \\ &= \tan.^2 D + \left( \frac{\text{tan. } d}{\sin. a} + \frac{\text{tan. } D}{\tan. a} \right)^2 \end{aligned}$$

We must now find the numerical value of the second member. Using logarithmic sines, cosines, tangents, &c., we must diminish the indices by 10, because the equation refers to radius unity.

Log.  $\tan. D = -1.561485$ ;  $\tan.^2 D = -1.122970 = 0.132730$  num

Log.  $\tan. d$        $-1.260623$       log.  $\tan. D$        $-1.561485$

$\sin. a$        $-1.603233$             $\tan. a$        $-1.641318$

$0.454349$        $-1.657390$        $0.832083$        $-1.920167$

$0.454349 + 0.832083 = 1.286432$        $(1.286432)^2 = 1.654906$

Whence,       $\cot.^2 L = 0.132730 + 1.654906 = 1.787636$

Square root,       $\cot. L = 1.337025$

Taking the log. of this number, increasing its index by 10 will give the log. cot. in our tables.

Log.  $1.337025 = 0.126139 + 10. = 10.126139 = \cot. 36^\circ 47' 38''$ .

(8.) On the 14th of November, 1829, Menkar was ob-

served to rise  $48^m 3^s$  before Aldebaran : what was the latitude of the observer. *Ans.*  $39^\circ 33' 53''$  north.

The position of these two stars in the heavens, November 1829, were as follows :

Menkar, right ascension,  $2^h 53^m 21^s$ . Dec.  $3^\circ 24' 52''$  north.  
 Aldebaran, "  $4^h 26^m 7^s$ . Dec.  $16^\circ 19' 31''$  north.

Aldebaran passes the meridian  $1^h 32^m 46^s$  after Menkar. Now let  $M$  represent the time Menkar was on the meridian, then  $M + 1^h 32^m 46^s$  represents the time Aldebaran was on the meridian. Also, let  $x =$  the arc of ascensional difference corresponding to the latitude and the star Menkar, and  $y$  that of the star Aldebaran.

Then,  $M - \left(6 + \frac{x}{15}\right) =$  the time Menkar rose.

And,  $M + 1^h 32^m 46^s - \left(6 + \frac{y}{15}\right) =$  the time Aldebaran rose.

Subtracting the upper from the lower, the difference must be  $48^m 3^s$ ; that is,

$$1^h + 32^m 46^s - \frac{y}{15} + \frac{x}{15} = 48^m 3^s$$

Whence,  $\frac{x}{15} - \frac{y}{15} = -44^m 43^s = -0.74527$ .

That is,  $1^h$  being the unit,  $44^m 43^s = 0.74527$  of an hour, and multiplying by 15, we shall have as many degrees of arc as we have units ; therefore,

$$x - y = -(0.74527)15 = -11^\circ 10' 45'' = -a$$

$$x = y - a$$

$$\sin. x = \sin. y \cos. a - \cos. y \sin. a \quad (1)$$

Put  $d = 3^\circ 24' 52''$ ,  $D = 16^\circ 19' 31''$ , and  $L =$  the required latitude. Then by scholium to the 4th example,

$$\sin. x = \tan. d \tan. L. \quad \sin. y = \tan. D \tan. L.$$

The values of  $\sin. x$  and  $\sin. y$ , substituted in (1), give

$$\text{Tan. } d.\text{tan. } L = \cos. a.\text{tan. } D.\text{tan. } L - \cos. y.\sin. a \quad (2)$$

But,  $\sin.^2 y = \tan.^2 D.\tan.^2 L$ , and  $1 - \sin.^2 y = 1 - \tan.^2 D.\tan.^2 L$ .

$$\text{Or,} \quad \cos.^2 y = 1 - \tan.^2 D.\tan.^2 L.$$

$$\text{Or,} \quad \cos. y = \sqrt{1 - \tan.^2 D.\tan.^2 L}.$$

By substituting this value of  $\cos. y$  in (2) and transposing, we find

$$\sin. a \sqrt{1 - \tan.^2 D.\tan.^2 L} = (\cos. a.\text{tan. } D. - \text{tan. } d)\text{tan. } L:$$

Dividing by  $\sin. a$ , and observing that  $\frac{\cos. a}{\sin. a} = \frac{1}{\tan. a}$ , we have

$$\sqrt{1 - \tan.^2 D.\tan.^2 L} = \left( \frac{\tan. D}{\tan. a} - \frac{\text{tan. } d}{\sin. a} \right) \text{tan. } L.$$

Squaring and dividing by  $\tan.^2 L$ , and at the same time observing that  $\frac{1}{\tan. L} = \cot. L$ , and we shall have

$$\cot.^2 L - \tan.^2 D = \left( \frac{\tan. D}{\tan. a} - \frac{\text{tan. } d}{\sin. a} \right)^2$$

We will now find the numerical value of the known quantities.

$$\text{Log. tan. } D \quad -1.466718 \quad \text{log. tan. } d \quad -2.775712$$

$$\text{Log. tan. } a \quad -1.295849 \quad \text{log. sin. } a \quad -1.287530$$

$$\text{Log. } 1.482072 \quad \overline{0.170869} \quad \text{log. } 0.307738 \quad -\overline{1.488182}$$

$$\text{Tan.}^2 D = 0.085790 \quad 1.482072 - 0.307738 = 1.174334$$

$$\text{Whence,} \quad \cot.^2 L - 0.085790 = (1.174334)^2.$$

$$\text{Or,} \quad \cot.^2 L = 1.464849.$$

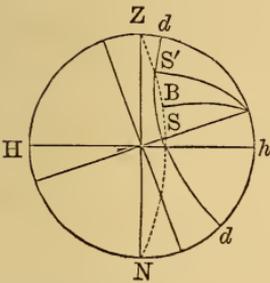
$$\cot. L = 1.210309.$$

$$\text{Log. cot. } L + 10 = 10.082896 = \cot. 39^\circ 33' 53''. \quad \text{Ans.}$$

(9.) In latitude  $16^\circ 40'$  north, when the sun's declination was  $23^\circ 18'$  north, I observed him twice, in the same fore-

noon, bearing north  $68^{\circ} 30'$  east. Required the time of observation, and his altitude at each time.

*Ans.* Times,  $6^h 15^m 40^s$ , A.M., and  $10^h 32^m 48^s$ , A.M., altitude,  $9^{\circ} 59' 33''$ , and  $68^{\circ} 29' 43''$ .



Let  $Z$  be the zenith,  $P$  the north pole, and the curve  $dd$  be the parallel of the sun's declination along which it appears to revolve. Make the angle  $PZS'$  equal to  $68^{\circ} 30'$ ; then the sun was at  $S$  at the time of the first observation, and at  $S'$  at the time of

the second.

In the spherical  $\triangle PZS'$  there is given  $PZ$ ,  $PS'$  and the angle  $PZS'$ ; also, in the  $\triangle PZS$  there is given  $PZ$ ,  $PS$ , and the angle  $PZS$ . Observe that  $PSS'$  is an *isosceles*  $\triangle$ .

Describe the meridian  $PB$  bisecting the angle  $S'PS$ , and then we have three right angled spherical triangles,  $BPS$ ,  $BPS'$ , and  $BPZ$ ; taking the last, we have the following proportion:

$$R. : \sin. PZ :: \sin. PZB : \sin. PB.$$

That is,

$$R. : \cos. 16^{\circ} 40' :: \sin. 68^{\circ} 30' : \sin. PB = \sin. 63^{\circ} 2' 27''.$$

To find  $ZB$ , we take the following proportion, (see page 185, Observation 1, Robinson's Geometry),

$$R. : \cos. ZB :: \cos. BP : \cos. PZ.$$

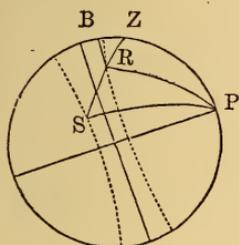
That is,  $R. : \cos. ZB :: \cos. 63^{\circ} 2' 27'' : \sin. 16^{\circ} 40'.$

$R. \sin. 16^{\circ} 40'$	19.457584
Cos. $63^{\circ} 2' 27''$	9.656439
Cos. $ZB, 50^{\circ} 45' 22''$	<hr style="width: 100%; border: 0.5px solid black;"/> 9.801145



The positions of the stars in the heavens were

Regulus, right ascension,  $10^h 0^m 10^s$ . Dec.  $12^\circ 43'$  north.  
 Spica, " "  $13^h 17^m 2^s$ . Dec.  $10^\circ 21' 20''$  south.



Let  $R$  be the position of Regulus,  $S$  the position of Spica,  $P$  the pole, and  $Z$  the zenith.

Then the side  $PS=100^\circ 21' 20''$ ,  $PR=77^\circ 17'$ , and the angle  $RPS=3^h 16^m 52^s$ , converted into degrees; that

is,  $RPS=49^\circ 13'$ .

One hour and ten minutes reduced to arc, give  $17^\circ 30'$ ; but the stars revolve according to *siderial*, not solar time, and to reduce solar to siderial arc, we must increase it by about its  $\frac{1}{365}$ th part; this gives about  $3'$  to add to  $17^\circ 30'$ , making  $17^\circ 33'$  for the angle  $ZPR$ . Our ultimate object is to find  $PZ$ , the complement of the latitude.

In the  $\triangle PRS$ , we have the two sides  $PR, PS$ , and the included angle  $P$ , from which we must find  $RS$  and the angle  $SRP$ , and we can let a perpendicular fall from  $R$  on to the side  $PS$  and solve it by the usual way; but to show that a wide field is open for a bold operator, we will put the unknown arc  $RS=x$ , the side opposite  $R=r$ , and opposite  $S=s$ , and apply one of the equations in formula ( $S$ ), page 191, Robinson's Geometry.

That is, 
$$\cos. P = \frac{\cos. x - \cos. r. \cos. s}{\sin. r. \sin. s}.$$

Whence, 
$$\cos. P. \sin. r. \sin. s + \cos. r. \cos. s = \cos. x.$$

We now apply this equation, recollecting that radius is unity, which will require us to diminish indices of the logarithms by 10.

$$\begin{array}{rclcl}
 \text{Cos. } P = \cos. 49^\circ 13' & -1.815046 & & & \\
 \text{Sin. } r = \sin. 100^\circ 21' 20'' & -1.992868 & -\cos. & -1.254683^* & \\
 \text{Sin. } s = \sin. 77^\circ 17' & -1.989214 & \cos. & -1.342679 & \\
 & 0.62680 & -1.797128 & .03957 & -2.597362
 \end{array}$$

$$\text{Cos. } x = 0.6268 - 0.03957 = .58723.$$

Whence, by the table of natural cosines, we find

$$x = 54^\circ 2' 20''.$$

To find the angle  $SRP$  or  $ZRP$ , we have

$$\text{Sin. } 54^\circ 2' 20'' : \text{sin. } 49^\circ 13' :: \text{sin. } 100^\circ 21' 20'' : \text{sin. } ZRP.$$

$$\text{Whence, } ZRP = 66^\circ 57' 37''.$$

Let fall the perpendicular  $RB$  on  $PZ$  produced, then the right angled spherical  $\triangle PBR$  gives this proportion,

$$R. : \text{sin. } 77^\circ 17' :: \text{sin. } 17^\circ 33' : \text{sin. } RB = \text{sin. } 17^\circ 6' 22''.$$

To find  $PB$ , we have

$$R. : \cos. PB :: \cos. 17^\circ 6' 22'' : \cos. 77^\circ 17'.$$

$$\text{Whence, } PB = 76^\circ 41'.$$

Now, to find the angle  $BRP$ , we have

$$\text{Sin. } 77^\circ 17' : R. :: \text{sin. } 76^\circ 41' : \text{sin. } BRP = \text{sin. } 86^\circ 1'.$$

From  $PRB$  take  $PRZ$ , and  $ZRB$  will remain; that is,

$$\text{From } 86^\circ 1' \text{ take } 66^\circ 57' 37'', \text{ and } ZRB = 19^\circ 3' 23''.$$

By the application of equation (12), we find that

$$R. \text{sin. } 17^\circ 6' 22'' = \tan. BZ. \cot. 19^\circ 3' 23''.$$

$$\text{Whence, } BZ = 5^\circ 48'$$

$$\text{And, } PZ = 76^\circ 41' - 5^\circ 48' = 70^\circ 53'.$$

The complement of  $70^\circ 53'$  is  $19^\circ 7'$ , the latitude sought.

By this example we perceive that by the means of a meridian line, a good watch, and a plumb line, any person having a knowledge of spherical trigonometry, and having a catalogue of the stars at hand, can determine his latitude by observation.

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\* Observe that  $r$  is greater than  $90^\circ$ , its cosine is therefore, negative in value, rendering the product,  $\cos. r. \cos. s$ , or .03957, negative.

# KEY TO ROBINSON'S ANALYTICAL GEOMETRY.

## CHAPTER I. STRAIGHT LINES.

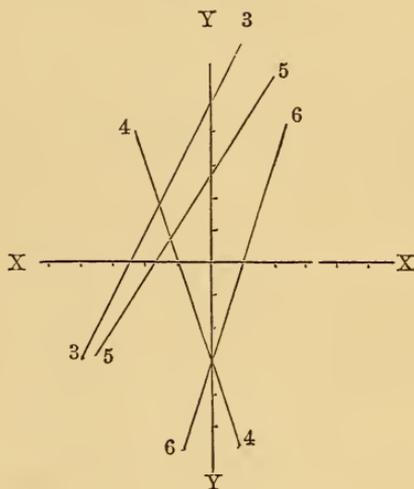
(PAGE 105.)

Ex. 3.  $y=2x+5$ .

Draw the rectangular coordinate axes,  $XX$ ,  $YY'$ .

Then, when  $y=0$ , the equation, give  $x=-\frac{5}{2}=-2\frac{1}{2}$ , and when  $x=0$ ,  $y=5$ ; hence the line represented by the equation cuts the axis of  $X$  at the distance  $-2\frac{1}{2}$ , and the axis of  $Y$  at the distance  $+5$  from the origin or zero point.

Assuming any convenient unit, lay it off on the axis several times, in both the positive and negative directions, and draw a line cutting the axis of  $X$  at  $-2\frac{1}{2}$ , and the axis of  $Y$  at  $+5$  from the origin. This will be  $(3, 3)$ , the line required, since two points determine the position of a line.



Ex. 4.  $y=-3x-3$ .

Making  $y=0$  in this equation gives  $x=-1$ , and  $x=0$  gives  $y=-3$ ; hence, this line cuts the axis of  $X$  at  $-1$  and

the axis of  $Y$  at  $-3$ . Laying off these distances on their respective axes, and drawing through the points thus determined the line (4, 4) it is that required.

Ex. 5.  $2y=3x+5$ , or  $y=\frac{3}{2}x+\frac{5}{2}$ .

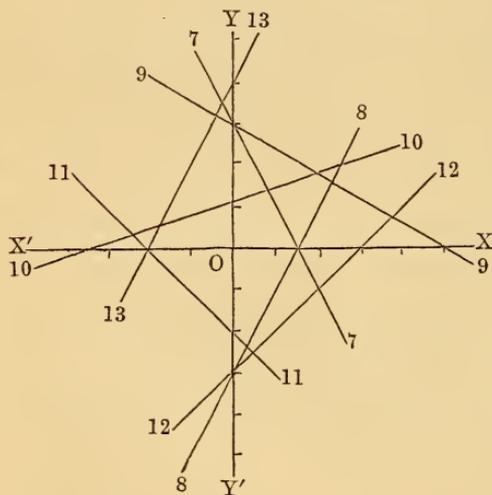
In this equation  $y=0$  makes  $x=-\frac{5}{3}=-1\frac{2}{3}$ , and  $x=0$  makes  $y=\frac{5}{2}=2\frac{1}{2}$ . The line (5, 5), drawn through the points in the axis, given by these values of  $x$  and  $y$ , is that represented by the equation.

Ex. 6.  $y=4x-3$ .

The values of  $x$  and  $y$  given by this equation, by making first,  $y=0$ , and then  $x=0$ , are  $x=\frac{3}{4}$ ,  $y=-3$ . The line is (6, 6).

The lines of which equations 4 and 6 are the equations both intersect the axis of  $Y$  at the distance  $-3$  from the origin.

Ex. 7.  $y=-2x+3$ .



Making in this equation  $y=0$  and  $x=0$ , successively, we find  $x=\frac{3}{2}=1\frac{1}{2}$ ,  $y=3$ , and the required line is (7, 7).

Ex. 8.  $y=2x-3$ .

Proceeding with this as with the preceding equation, we get  $x=1\frac{1}{2}$ ,  $y=-3$ ; and (8, 8) is the line responding to this equation.

In the triangle  $CAB$  since  $OA$  and  $OB$  are each equal 3,

$AB$ , the base, is equal to 6, and because  $OC$  is perpendicular to the base and bisects it the triangle is isosceles.  $OC$  is equal to  $1\frac{1}{2}$  by the construction.

Ex. 9.  $3x+5y-15=0$ , or  $y=-\frac{3}{5}x+3$ .

The suppositions  $y=0$  and  $x=0$ , made successively, give  $x=5$ ,  $y=3$ , and the line (9, 9) drawn through the point +5 on the axis of  $X$ , and the point +3 on the axis of  $Y$  responds to the equation.

Ex. 10.  $2x-6y+7=0$ , or  $y=\frac{1}{3}x+\frac{7}{6}$ .

From this we find the intersections of the line with the axes of  $X$  and  $Y$  respectively to be at the distances  $-\frac{7}{2}$  and  $+\frac{7}{6}$  from the origin, and (10, 10) is the required line.

Ex. 11.  $x+y+2=0$ , or  $y=-x-2$ .

This line cuts both the axes at the distance  $-2$  from the origin and makes with each an angle of  $45^\circ$ . It is the line (11, 11).

Ex. 12.  $-x+y+3=0$ , or  $y=x-3$ .

This line cuts both axes at the distance of 3 units from the origin; the axis of  $X$  at +3, the axis of  $Y$  at  $-3$ , it therefore makes with each an angle of  $45^\circ$ . It is the line (12, 12).

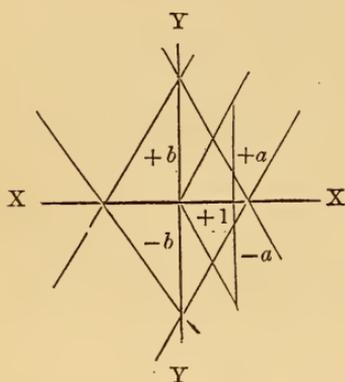
Ex. 13.  $2x-y+4=0$ , or  $y=2x+4$ .

In this  $y=0$  gives  $x=-2$ , and  $x=0$  gives  $y=4$ ; hence, the line (13, 13) which cuts the axis of  $X$  at  $-2$ , and the axis of  $Y$  at +4 is that which responds to the equation.

If we solve any equation of the first degree between two variables with reference to one of the variables it will take either the form  $y=ax+b$ , or  $x=a'y+b'$ . Now  $a$  denotes the tangent of the angle that the line makes with the axis of  $X$ ,

and  $a'$  denotes the tangent of the angle the same line makes with the axis of  $Y$ , while  $b$  is the distance from the origin to the point in which the line cuts the axis of  $Y$ , and  $b'$  the distance from the origin to the point in which the line cuts the axis of  $X$ .

To construct this line we may draw through the origin a line making with the axis of  $X$  an angle having  $a$  for its tangent, and then the line drawn parallel to this through the point on the axis of  $Y$  at the distance  $b$  from the origin will be that represented by the equation.



To make this construction assume any convenient unit of measure, and lay it off on the axis of  $X$  from the origin to the right, that is, in the positive direction. Through the extremity of this unit draw a parallel to the axis of  $Y$ , and mark off on this parallel from the axis of  $X$  the distance  $a$

units, above the axis of  $X$  if  $a$  is positive, below if it is negative. The line which connects the origin with the point thus determined on the parallel will make with the axis of  $X$  an angle of which  $a$  is the tangent, and it is, therefore, parallel to the required line. If, then, we lay off the distance  $b$  units on the axis of  $Y$ , above, or below the origin, according as  $b$  is positive or negative, and through the extremity of this line we draw a line parallel to that passing through the origin we shall have the required line.

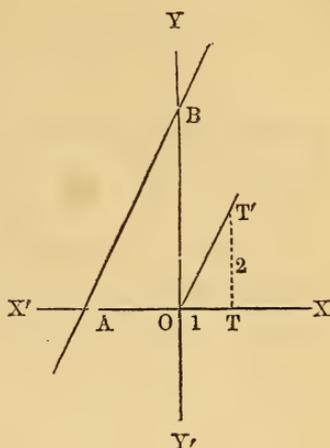
Thus, to construct the line represented by the equation  $2x - y + 4 = 0$  (example 13), we solve it with reference to  $y$  and get  $y = 2x + 4$ .

We lay off the distance  $OT =$  unity on the axis of  $X$  from

the origin to the right and on the parallel to the axis of  $Y$ , at this distance, mark off  $TT' =$  two units above the axis of  $X$ , because in this example  $a = +2$ .

Then make  $OB = 4$ , draw  $OT'$  and through  $B$  draw  $BA$  parallel to  $OT'$ , and the line given by the equation is constructed.

With this explanation the student will find no difficulty in applying this method of construction to all of the above examples.



(Page 108.)

To prove that equations (6) and (7) are different forms of the same equation.

By clearing equation (6) of fractions it becomes

$$yx'' - y'x'' - yx' + y'x' = y''x - y'x - y''x' + y'x'.$$

Canceling and transposing we find

$$yx'' - yx' = y'x'' + y''x - y'x - y''x'. \quad (8)$$

Equation (7) treated in the same way gives, first,

$$yx'' - y''x'' - yx' + y''x' = y''x - y'x - y''x' + y'x'.$$

And finally

$$yx'' - yx' = y'x'' + y''x - y'x - y''x'. \quad (9)$$

Since equations (8) and (9) are the same equations, (6) and (7) from which they were derived must represent the same line.

Equation (6) being

$$y - y' = \frac{y'' - y'}{x'' - x'} (x - x')$$

the suppositions  $y = y'$ , and  $x = x'$  reduces both members to zero; hence the line passes through the first point of which the co-ordinates are  $y'$  and  $x'$ .

If in this equation we suppose  $y=y''$  and  $x=x''$  it becomes

$$y''-y'=\frac{y''-y'}{x''-x'}(x''-x').$$

Dividing both members of this last equation by  $y''-y'$  and clearing of fractions it reduces to

$$x''-x'=x''-x' \text{ or } 0=0.$$

Therefore, the co-ordinates of both points when substituted for the variables  $x$  and  $y$  in equation (6) satisfy that equation, which is the condition that the line shall pass through these points.

(Page 109.)

Ex. 2. Placing in the equation

$$y-y'=\frac{y''-y'}{x''-x'}(x-x') \quad (\alpha)$$

for  $x'$ ,  $x''$ , and  $y'$ ,  $y''$  their values, this equation becomes

$$y+1=\frac{-\frac{1}{6}+1}{4\frac{1}{2}+4}(x+4).$$

Reducing

$$y+1=-\frac{4}{5}x-\frac{1}{5}.$$

Or,

$$y=-\frac{4}{5}x-1\frac{1}{5}.$$

Ex. 3. Making the substitutions in equation ( $\alpha$ ), (last example) for  $x'$ ,  $x''$ , and  $y'$ ,  $y''$ , it becomes

$$y-5=\frac{3-5}{-3-6}(x-6),$$

Or,

$$y-5=\frac{2}{9}x-\frac{1}{9}.$$

Whence,

$$y=\frac{2}{9}x+3\frac{2}{9}.$$

(Page 112).

Ex. 2. The equations

$$2y=5x+8$$

$$3y=-2x+6,$$

become by dividing through by the coefficients of  $y$ ,

$$y = \frac{5}{2}x + 4$$

$$y = -\frac{2}{3}x + 2.$$

Making  $a = \frac{5}{2}$  and  $a' = -\frac{2}{3}$  the formula.

$$m = \frac{a' - a}{1 + aa'}$$

Becomes

$$m = \frac{-\frac{2}{3} - \frac{5}{2}}{1 - \frac{10}{6}} = 4\frac{3}{4} = 4.75.$$

(Page 114.)

Ex 1. By making  $a = -2$ ,  $a' = 5$ ,  $b = 1$  and  $b' = 10$ , in the formula

$$x_1 = -\frac{b - b'}{a - a'} \text{ it becomes}$$

$$x_1 = -\frac{1 - 10}{-2 - 5} = -\frac{9}{7}.$$

The same substitutions in

$$y_1 = \frac{a'b - ab'}{a' - a} \text{ give}$$

$$y_1 = \frac{5 + 20}{5 + 2} = \frac{25}{7} = 3\frac{4}{7}.$$

(Page 115.)

Ex. 1. In the formula for the perpendicular,

$$\text{Per.} = \pm \frac{b + ax' - y'}{\sqrt{a^2 + 1}},$$

we must make  $a = 3$ ,  $b = -10$ ,  $x' = 4$ , and  $y' = 5$ . These values being substituted, we have

$$\text{Per.} = \pm \frac{-10 + 12 - 5}{\sqrt{3^2 + 1}} = \pm \frac{-3}{\sqrt{10}}.$$

Multiplying the numerator and denominator of this by

$\sqrt{10}$  and passing the 3 under the radical by squaring, we have finally

$$\text{Perpendicular} = \frac{1}{10} \sqrt{90}.$$

Ex. 2. Here  $a = -5$ ,  $b = -15$ ,  $x' = 4$ ,  $y' = 5$ , and by placing these in the formula for the perpendicular, we find

$$\begin{aligned} \text{Per.} &= \pm \frac{-15 - 20 - 5}{\sqrt{5^2 + 1}} = \pm \frac{-40}{\sqrt{26}} \\ &= \frac{1}{26} \times 40\sqrt{26} = \frac{203}{26} = 7.84+. \end{aligned}$$

(Page 118.)

Ex. 3. Dividing both of the equations

$$3y + 5x = 4$$

$$2y = 3x + 4,$$

through by  $y$  and transposing, they become

$$y = -\frac{5}{3}x + \frac{4}{3}$$

$$y = \frac{3}{2}x + 2.$$

Here  $a = -\frac{5}{3}$ ,  $a' = \frac{3}{2}$ ,  $b = \frac{4}{3}$ ,  $b' = 2$ . Placing these values in the formula

$$\begin{aligned} x_1 &= -\left(\frac{b-b'}{a-a'}\right), \text{ we get } x_1 = -\frac{\frac{4}{3}-2}{-\frac{5}{3}-\frac{3}{2}} = -\frac{-\frac{2}{3}}{-\frac{19}{6}-\frac{9}{6}} \\ &= -\frac{2 \times 6}{3 \times 19} = -\frac{4}{19}. \end{aligned}$$

Making the same substitutions in the formula

$$y_1 = \frac{a'b - ab'}{a' - a} \text{ we get } y_1 = \frac{\frac{3}{2} \times \frac{4}{3} + \frac{5}{3} \times 2}{\frac{3}{2} + \frac{5}{3}} = \frac{2 + \frac{10}{3}}{\frac{9}{6} + \frac{10}{6}} = \frac{\frac{16}{3}}{\frac{19}{6}} = \frac{32}{19}.$$

To get the natural tangent of  $30^\circ$  we have

$$\text{Tan. } 30^\circ = \frac{\sin. 30^\circ}{\cos. 30^\circ} = \frac{.50000}{.86603} = .5773.$$

These values of  $x_1$ ,  $y_1$ , and  $\tan. 30^\circ$  placed in the equation

$$y - y_1 = \tan. 30^\circ(x - x_1) \text{ give}$$

$$y - \frac{32}{19} = 0.5773(x + \frac{4}{19}).$$

(Page 119.)

Ex. 4. By transposing and dividing through by the coefficients of  $y$ , the equations

$$2y - 3x = -1$$

$$2y + 3x = 3.$$

Become

$$y = \frac{3}{2}x - \frac{1}{2}.$$

$$y = -\frac{3}{2}x + \frac{3}{2}.$$

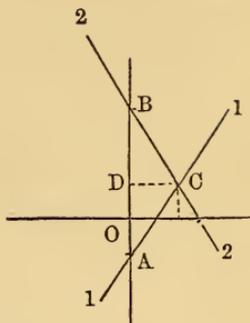
Whence  $a = \frac{3}{2}$ ,  $a' = -\frac{3}{2}$ ,  $b = -\frac{1}{2}$ ,  $b' = \frac{3}{2}$ .

These values put in the formulæ

$$x_1 = \frac{b-b'}{a'-a}, \quad y_1 = \frac{a'b-ab'}{a'-a}, \quad \text{give } x_1 = \frac{2}{3}, \quad y_1 = \frac{1}{2}.$$

The point  $C$  in which the lines intersect may then be constructed. Since the first line (1, 1), intersects the axis of  $Y$  at the distance  $-\frac{1}{2}$ , and the second line (2, 2) intersects it at the distance  $\frac{3}{2}$ , the distance  $AB$  is 2; and because the lines (1, 1) and (2, 2) are equally inclined to the axis of  $X$  on opposite sides, they are also equally inclined to the axis of  $Y$  on opposite sides; hence, the triangle  $ABC$  is isosceles. The side  $AC = BC = \sqrt{1 + (\frac{2}{3})^2} = \sqrt{\frac{13}{9}} = \frac{1}{3} \sqrt{13} = \frac{3.605+}{3} = 1.201+.$

The perpendicular,  $CD$ , of the triangle is  $\frac{2}{3}$ ; hence the area of the triangle is  $= \frac{1}{2} \times 2 = 0.66+.$



(Page 119.)

Ex. 5. The equations

$$-2\frac{1}{5}y + 3\frac{1}{2}x = -2\frac{1}{4},$$

$$2\frac{2}{5}y - \frac{2}{3}x = 4$$

by transposing and dividing through by the coefficients of  $y$  become

$$y = \frac{3}{2}\frac{5}{2}x + \frac{4}{4}\frac{5}{4},$$

$$y = \frac{5}{1}\frac{5}{8}x + \frac{5}{2}.$$

Whence  $a = \frac{3\frac{5}{2}}{2}$ ,  $a' = \frac{5}{1\frac{5}{8}}$ ,  $b = \frac{4\frac{5}{4}}{4}$ ,  $b' = \frac{5}{3}$ ,  
 substituting these in the formula for  $x_1$  and  $y_1$ , we get

$$x_1 = -\frac{\frac{4\frac{5}{4}}{4} - \frac{5}{3}}{\frac{3\frac{5}{2}}{2} - \frac{5}{1\frac{5}{8}}} = -\frac{-\frac{9\frac{5}{2}}{1\frac{3}{2}}}{\frac{5\frac{2}{9}}{6}} = \frac{3 \times 17}{104} = \frac{51}{104} = 0.49,$$

$$\text{and } y_1 = \frac{\frac{5}{1\frac{5}{8}} \times \frac{4\frac{5}{4}}{4} - \frac{3\frac{5}{2}}{2} \times \frac{5}{3}}{\frac{5}{1\frac{5}{8}} - \frac{3\frac{5}{2}}{2}} = \frac{\frac{2\frac{5}{8}}{8} - \frac{1\frac{7}{6}}{6}}{-\frac{1\frac{3}{9}}{9}} = 1.8.$$

The form of the equation of a line passing through two given points is

$$y - y'' = \frac{y'' - y_1}{x'' - x_1} (x - x''),$$

and this will become the equation of the required line when we make  $x_1 = 0.49$ ,  $y_1 = 1.8$ ,  $x'' = 3$ ,  $y'' = 0$ . In this case the expression  $\frac{y'' - y_1}{x'' - x_1}$  reduces to  $-0.7171 +$ .

Placing these values in the above equation we get

$$y = -0.7171x + 0.7171 \times 3 = -0.7171x + 2.1523.$$

And substituting the values of the two points in the formula.

$$D = \sqrt{(x'' - x_1)^2 + (y'' - y_1)^2}.$$

We have

$$D = \sqrt{(2.51)^2 + (1.8)^2}.$$

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## CHAPTER II.

### THE CIRCLE.

(Page 139.)

Ex. 1. In all these examples the equation to be referred to is

$$y^2 \mp cy = \frac{R^2 - c^2}{2}$$

Comparing the equation  $x^2 + 11x = 80$  with this, we have

$c=11$ ,  $\frac{R^2-c^2}{2}=80$ ; whence  $R^2=160+121=281$ .  $R=\sqrt{281}$   
 $=16.76$ , with which we proceed as explained in the text.

Ex. 2.  $x^2-3x=28$ .

Here  $c=-3$ ,  $\frac{R^2-c^2}{2}=28$ ; whence  $R^2=56+9=65$ ,  $R$   
 $=\sqrt{65}=8.06$ .

Ex. 3.  $x^2-x=2$ .

In this  $c=-1$ ,  $\frac{R^2-c^2}{2}=2$ ,  $R^2=5$ ,  $R=\sqrt{5}=2.23$ .

Ex. 4.  $x^2-12x=-32$ .

In this  $c=-12$ ,  $\frac{R^2-c^2}{2}=-32$ ,  $R^2=80$ ,  $R=8.94$ .

Ex. 5.  $x^2-12x=-36$ .

This gives,  $c=-12$ ,  $\frac{R^2-c^2}{2}=-36$ .  $R^2=72$ ,  $R=8.48$ .

Ex. 6.  $x^2-12x=-38$ .

Whence  $c=-12$ ,  $\frac{R^2-c^2}{2}=-38$ ,  $R^2=68$ ,  $R=8.24$ .

Ex. 7.  $x^2+6x=-10$ .

Whence  $c=6$ ,  $\frac{R^2-c^2}{2}=-10$ ,  $R^2=16$ ,  $R=4$ .

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## CHAPTER IV.

### THE PARABOLA.

(Page 187.)

Ex. 3.  $x^2-\frac{2}{11}x=8$ .

Comparing this with the equation

$$R^2+2bR=2c-b^2$$

we see that  $b=-\frac{1}{11}$ ,  $2c-b^2=8$ ; whence  $2c=8+\frac{1}{11}=8\frac{1}{11}$ ,

$c = \frac{2}{3} \frac{6}{4} \frac{2}{2} = 4+$ . We construct the pole by laying off the distance 4 on the axis to the right, for the abscissa of the pole, and the distance  $-\frac{1}{11}$  for the ordinate.

The values of  $x$  will be found to be  $+2.9+$  and  $-2.7+$ .

Ex. 4.  $\frac{3}{4}x^2 + \frac{3}{5}x = \frac{7}{11}$ , or  $x^2 + \frac{4}{5}x = \frac{2}{3} \frac{2}{3}$ .

Here  $2b = \frac{4}{5}$ ,  $b = \frac{2}{5}$ ,  $2c - b^2 = \frac{2}{3} \frac{2}{3}$ ,  $2c = \frac{2}{3} \frac{2}{3} + \frac{4}{5}$ ,  $c = .504+$ . If the pole be constructed with these coordinates, the distances from the pole, above and below the axis, to the intersection of the perpendicular radius vector with the curve are  $.6+$  and  $1.4+$ .

Ex. 5.  $\frac{1}{4}y^2 - \frac{1}{6}y = 2$ , or  $y^2 - \frac{2}{3}y = 8$ .

We find for this  $2b = -\frac{2}{3}$ ,  $b = -\frac{1}{3}$ ,  $2c - b^2 = 8$ ,  $2c = 8 + \frac{1}{9}$ , whence  $c = 8.1+$ . Constructing the pole with these coördinates, and drawing the perpendicular radius vector, we shall find the distances from the pole to the intersections with the curve to be  $=3.17+$  and  $-2.5+$ .

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## CHAPTER VI.

### INTERPRETATION OF EQUATIONS.

(Page 223.)

Ex. 1. We find the abscissas of the vertices of the diameter whose equation is  $y = -x$ , by placing the quantity  $-2x(x-2)$  under the radical in the general value of  $y$  equal to zero; that is we make

$$-2x(x-2) = 0$$

which gives the two values  $x' = 0$ ,  $x'' = 2$ . These values of  $x$  substituted in the equation of the diameter  $y = -x$ , give  $y' = 0$ ,  $y'' = -2$ ; hence  $x' = 0$ ,  $y' = 0$  are the coordinates of one vertex, and  $x'' = 2$ ,  $y'' = -2$  are those of the other.

The abscissa of the center  $x=1$ , and giving  $x$  this value in the proposed equation it reduces to

$$y^2 + 2y + 3 - 4 = 0$$

Or,  $y^2 + 2y = 1$

Whence,  $y = -1 \pm \sqrt{2} = -1 \pm 1.41 +$ ,

That is  $y = +.41 +$  and  $y = -2.41 +$ .

(Page 236.)

Ex. 6. In the equation

$$y^2 - 2xy - x^2 - 2y + 2x + 3 = 0,$$

we see, by comparing it with the general equation that  $A=1$ ,  $B=-2$ ,  $C=-1$ ; hence  $B^2 - 4AC > 0$  and the equation represents an hyperbola.

Solving the equation with reference to  $y$ , we find

$$y = x + 1 \pm \sqrt{2(x^2 - 1)}.$$

Placing the quantity under the radical in this value of  $y$ , equal to zero we have  $x = +1$ , and  $x = -1$  for the abscissas of the vertices of the diameter which bisects the chords parallel to the axis of  $Y$ ; hence the axis of  $Y$  bisects this diameter and is the conjugate to it.

By making  $y$  equal to zero in the proposed equation it becomes

$$x^2 - 2x = 3$$

Whence,  $x = 1 \pm \sqrt{3+1} = 1 \pm 2$ ,  $x = +3$ , or  $-1$ .

Ex. 7. Comparing the equation

$$y^2 - 2xy + 2x^2 - 2x + 4 = 0$$

with the general equation, we find,

$$A=1, B=-2, C=2; \text{ therefore } B^2 - 4AC < 0,$$

and the analytical condition for the ellipse is satisfied; but since the equation can be put under the form

$$(y-x)^2 + (x-1)^2 + 3 = 0$$

we see that all real values of  $x$  and  $y$  will make the first

number the sum of three positive quantities, and hence such values cannot satisfy the equation.

(Page 237.)

Ex. 8.  $y^2 - 2xy + x^2 + x = 0.$

This represents the parabola, because  $A=1, B=-2, C=1$  and, therefore,  $B^2 - 4AC=0$ . The curve passes through the origin, since the equation contains no absolute term and is therefore satisfied by  $x=0, y=0$ .

Solving the equation in respect to  $y$ , we find

$$y = x \pm \sqrt{-x},$$

from which we conclude that all positive values of  $x$  will give imaginary values to  $y$ , or that the curve does not extend in the direction of  $x$  positive.

If the equation be solved with reference to  $x$  we shall find

$$x = \frac{2y-1}{2} \pm \sqrt{-y^2 + \frac{4y^2 - 4y + 1}{4}} = \frac{2y-1}{2} \pm \frac{1}{2} \sqrt{1-4y}.$$

The radical part of this value of  $x$  shows that any positive value of  $y$  greater than  $\frac{1}{4}$ , will render  $x$  imaginary, but that  $x$  will be real for all negative values of  $y$ . Hence the curve extends indefinitely in the direction of  $x$  and  $y$  negative. Substituting  $\frac{1}{4}$  for  $y$ , in the equation, it becomes

$$\frac{1}{16} - \frac{1}{2}x + x^2 + x = 0$$

Or,  $x^2 + \frac{1}{2}x + \frac{1}{16} = (x + \frac{1}{4})^2 = 0$

Whence,  $x + \frac{1}{4} = 0, x = -\frac{1}{4}.$

A line drawn through the point whose coördinates are  $y = +\frac{1}{4}, x = -\frac{1}{4}$ , parallel to the axis of  $X$ , will limit the curve in the direction of  $y$  positive; but it will be unlimited in the direction of  $y$  negative.

Ex. 9.  $y^2 - 2xy + x^2 - 2y - 1 = 0.$

The curve represented by this equation is the parabola.

Since we have for it,  $A=1$ ,  $B=-2$ ,  $C=1$ , and, therefore,  
 $B^2-4AC=0$ .

Transposing and factoring it becomes

$$y^2-2(x+1)y=-x^2+1,$$

Whence,

$$y=x+1 \pm \sqrt{-x^2+1+x^2+2x+1} = x+1 \pm \sqrt{2(x+1)}.$$

Also,  $x^2-2xy=-y^2+2y+1$ ,

Whence,  $x=y \pm \sqrt{2y+1}$ .

The value of  $y$  shows that all negative values of  $x$  greater, numerically, than  $-1$  will render  $y$  imaginary, and the value of  $x$  shows that all negative values of  $y$  greater, numerically, than  $-\frac{1}{2}$  will render  $x$  imaginary. The curve is therefore limited in the direction of the negative axes; but it is unlimited in the opposite direction.

Ex. 10.  $y^2-4xy+4x^2=0$ .

Since in this equation,  $A=1$ ,  $B=-4$ ,  $C=4$ , we have  $B^2-4AC=0$  which is the condition for the parabola. But we have also the further conditions (page 219),

$$BD-2AE=0, \quad D^2-4AF=0.$$

In consequence of the first of these conditions, the parabola would reduce to two parallel straight lines, and in consequence of the second, the two straight lines reduce to one, or coincide.

The first member of the equation is a perfect square, and we therefore have

$$(y-2x)^2=0, \quad \text{or } y=2x.$$

Hence the line to which the parabola reduces, passes through the origin of coördinates and makes, with the axis of  $X$ , an angle, having 2 for its natural tangent. The degrees and minutes of the angle are found thus

$$\text{Log. } 2 = 0.301030$$

$$\text{Log. } R = 10.000000$$

$$\text{Log. tangent } 63^\circ 26', \quad \underline{\underline{10.301030.}}$$

The complement of  $63^\circ 26'$  is  $26^\circ 34'$ , which is the angle the line makes with the axis of  $Y$ .

Ex. 11.  $y^2 - 2xy + 2x^2 - 2y + 2x = 0$ .

Here  $A=1$ ,  $B=-2$ ,  $C=2$ . Hence  $B^2 - 4AC < 0$ , and the curve is an ellipse.

Solving the equation with reference to  $y$ , we get

$$y = x + 1 \pm \sqrt{-x^2 + 1}.$$

Placing the quantity under the radical sign equal to zero, to get the abscissas of the vertices of the diameter whose equation is  $y = x + 1$ , we find

$$x = \pm 1, \text{ or } x = +1 \text{ and } x = -1.$$

The value of  $y$  shows that all substitutions for  $x$ , numerically greater than 1, positive or negative, will render  $y$  imaginary. Hence parallels to the axis of  $Y$ , drawn at a unit's distance from it on either side, will limit the curve.

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## CHAPTER VII.

### ON THE INTERSECTION OF LINES AND THE GEOMETRICAL SOLUTION OF EQUATIONS.

(PAGE 243.)

Ex. 3. Given  $y^2 - 48y = 128$ , to find the values of  $y$  by construction.

Making  $y = nz$  and substituting in the given equation it becomes

$$n^2 z^2 - 48nz = 128,$$

Whence, 
$$z^2 - \frac{48}{n^2} z = \frac{128}{n^2}.$$

If now, we assume  $n=4$ , this equation will reduce to

$$z^3 - 3z = 2,$$

and the construction of the values of  $z$  is the same as in example 2; that is  $z=2$ ,  $z=-1$ ,  $z=-1$ , and since  $y=nz$ , and  $n=4$  we have,

$$\text{Ans. } y = +8, -4, -4.$$

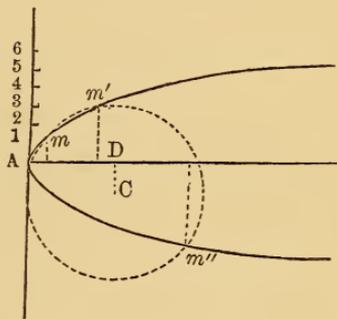
Ex. 4. Given  $y^3 - 13y = -12$ , to find the values of  $y$  by construction.

Comparing this equation with equation (G) page 241, we have

$$4 - 4a = -13, \quad 8b = -12,$$

$$\text{Whence, } a = 4\frac{1}{4}, \quad b = -1\frac{1}{2}.$$

Constructing the parabola with the distance from the focus to the directrix for the unit, we then lay off on its axis, in the positive direction, the distance  $AD = 4\frac{1}{4}$ , then



$DC = -1\frac{1}{2}$  will determine the center of the circle. The circumference described with  $C$  as a center and  $CA$  as a radius, cuts the parabola in the points  $m, m', m''$ , which are at the distances from the axis of the parabola  $+1, +3$ , and  $-4$ , respectively; hence

$$\text{Ans. } y = +1, +3, -4.$$

The different values of  $x$ , on page 246, corresponding to the assumed values 30.0388, 25 ---  $-5$ , &c., of  $y$  are thus found;

Equation (A) page 245, when  $y=30.0388$  becomes

$$x^3 = 13x + 12 = 30.0388,$$

$$\text{Or, } x^3 - 13x = 18.0388.$$

Whence,  $4 - 4a = -13, 8b = 18.0388$  (see eq. (G) page 241).

Therefore,  $a = 4\frac{1}{4}, b = 2.25485$ .

These values of  $a$  and  $b$  will give us the center of the circumference, the intersections of which with the parabola, determine the values of  $x$ .

So when  $y=25$  we shall have

$$x^3 - 13x = 13$$

$$4 - 4a = -13, \quad 8b = 13; \quad a = 4\frac{1}{4}, \quad b = 1\frac{5}{8},$$

and so on, the value of  $a$  being constantly equal to  $4\frac{1}{4}$ .

## CHAPTERS VIII., and IX.

### STRAIGHT LINES IN SPACE AND PLANES.

(Page 269.)

Ex. 1. What is the distance between two points in space of which the coördinates are

$$x=1, \quad y=-5, \quad z=-3; \quad x'=4, \quad y'=-4, \quad z'=1.$$

Substituting these values of  $x, y, z, x', y', z'$ , in the formula

$$D^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

found on page 254 (Prop. 6), and taking the square root we get

$$D = \sqrt{(3+2)^2 + (5+1)^2 + (-2-6)^2}$$

$$D = \sqrt{25+36+64} = \sqrt{125} = 11.180+.$$

*Ans.*

Ex. 2. Of which the coördinates are

$$x=1, \quad y=-5, \quad z=-3; \quad x'=4, \quad y'=-4, \quad z=1.$$

As in Example 1, substituting these values of the coördinates of the two points we find

$$D = \sqrt{(1-4)^2 + (-5+4)^2 + (-3-1)^2}$$

$$= \sqrt{9+1+16} = \sqrt{26}$$

$$= 5.098+ = 5\frac{1}{10} \text{ nearly.}$$

*Ans.*

Ex. 3. The equations of the projections of a straight line on the coördinate planes ( $xz$ ), ( $yz$ ), are

$$x=2z+1, \quad y=\frac{1}{3}z-2,$$

required the equation of the projection on the plane ( $xy$ ).

Multiplying the second equation through by 6, we have

$$6y=2z-12$$

which makes the coefficients of  $z$  the same in the two equations. Subtracting the first equation from this last, member from member, we get

$$6y-x=-13$$

Or, 
$$y=\frac{1}{6}x-2\frac{1}{6}. \quad \text{Ans.}$$

Ex. 4. The equations of the projections of a line on the coördinate planes ( $xy$ ) and ( $yz$ ), are

$$2y=x-5 \text{ and } 2y=z-4.$$

Required the equation of the projection on the plane ( $xz$ ).

By subtracting the second of these equations from the first, member from member, we have

$$0=x-z-1, \text{ or } x=z+1. \quad \text{Ans.}$$

Ex. 5. Required the equations of the three projections of a straight line, which passes through two points, whose coördinates are

$$x'=2, \quad y'=1, \quad z'=0; \text{ and } x''=-3, \quad y''=0, \quad z''=-1.$$

Placing these values of  $x'$ ,  $x''$ ,  $y'$ , &c., in the formulas on page 252 (Prop. 8), which are

$$x-x' = \frac{x''-x'}{z''-z'}(z-z'); \quad y-y' = \frac{y''-y'}{z''-z'}(z-z'),$$

the first becomes

$$x-2 = \frac{-3-2}{-1}z, \text{ or } x=5z+2,$$

and the second

$$y-1 = \frac{-1}{-1}z, \text{ or } y=z+1,$$

} *Answers.*

which are the equations of the projections of the required line on the planes  $(xz)$ ,  $(yz)$ .

We eliminate  $z$  from these equations by subtracting the first from the second after multiplying the second through by 5 :

Thus

$$5y = 5z + 5$$

$$x = 5z + 2$$

Subtract and  $5y - x = 5 - 2$ , or  $5y = x + 3$ . *Ans.*

Which is the equation of the projection of the line on the plane  $(xy)$ .

Ex. 6. Required the angle included between two lines whose equations are

$$\left. \begin{array}{l} x = 3z + 1 \\ y = 2z + 6 \end{array} \right\} \text{ of the 1st ; and } \left\{ \begin{array}{l} x = z + 2 \\ y = -z + 1 \end{array} \right\} \text{ of the 2d.}$$

Referring to the formula

$$\text{Cos. } V = \frac{1 + aa' + bb'}{\sqrt{1 + a^2 + b^2} \sqrt{1 + a'^2 + b'^2}}$$

found on page 257, Prop. 8, and comparing the coefficients of  $z$  in the given equations with those of the equations in that proposition, we have

$$a = 3, a' = 1 ; b = 2, b' = -1.$$

These values substituted in the formula give

$$\text{Cos. } V = \frac{1 + 3 - 2}{\sqrt{1 + 9 + 4} \sqrt{1 + 1 + 1}} = \frac{2}{\sqrt{42}} = \frac{1}{\sqrt{21}} \sqrt{42}.$$

$$\sqrt{42} = 6.4808 ; \text{ hence } \frac{1}{\sqrt{21}} \sqrt{42} = .308609.$$

That is

$$\text{Cos. } V = .308609.$$

The table gives .30846 for the cosine of  $72^\circ 2'$ , and .30874 for the cosine of  $72^\circ 1'$  ; hence we have the proposition

$$28 : 13 :: 60'' : x'' = 28'',$$

And  $V = 72^\circ 1' 28''$ . *Ans.*

Ex. 7. Find the angles made by the lines, designated in the preceding example, with the coördinate axes.

Formulas (5), (6) and (7), page 255, Prop. (7), are

$$\text{Cos. } X = \frac{a}{\pm \sqrt{1+a^2+b^2}}. \quad (5)$$

$$\text{Cos. } Y = \frac{b}{\pm \sqrt{1+a^2+b^2}}. \quad (6)$$

$$\text{Cos. } Z = \frac{1}{\pm \sqrt{1+a^2+b^2}}. \quad (7)$$

In these, to find the angles the first line makes with the coördinate axes, we must make  $a=3$ ,  $b=2$ .

These values placed in the above formulas give

$$\begin{aligned} \text{Cos. } X &= \frac{3}{\pm \sqrt{1+9+4}} = \frac{3}{\pm \sqrt{14}} \\ &= \frac{3\sqrt{14}}{14} = \frac{1}{4}\sqrt{126} = \frac{1}{4}(11.224+). \end{aligned}$$

Cos.  $X = .80179$ . By the table,  $X = 36^\circ 42'$ . *Ans.*

$$\text{Cos. } Y = \frac{2}{\pm \sqrt{14}} = \frac{1}{4}\sqrt{56} = \frac{1}{4}(7.4833+)$$

Cos.  $Y = .53452$ .  $y = 57^\circ 41' 20''$ . *Ans.*

$$\text{Cos. } Z = \frac{1}{\pm \sqrt{14}} = \frac{1}{4}\sqrt{14} = \frac{1}{4}(3.7416+)$$

Cos.  $Z = .26727$ .  $z = 74^\circ 29' 54''$ . *Ans.*

In like manner to find the angles that the second line makes with the coördinate axes, we must make  $a=a' = 1$ ,  $b=b' = -1$ . Formulas (5) and (7) then give

$$\text{Cos. } X = \text{cos. } Z = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3} = \frac{1}{3}(1.73205)$$

Cos.  $X = \text{cos. } Z = .57735$ ,  $X = Z = 54^\circ 44'$ . *Ans.*

Since  $b' = -1$ , the cosine of  $Y$  must be taken with a minus sign, if the cosines of  $X$  and  $Z$  are taken with the plus sign,

and the converse. But these cosines have the same numerical value ; therefore  $Y$  is the supplement of  $54^\circ 44'$  ; that is

$$Y = 125^\circ 16'. \quad \text{Ans.}$$

Ex. 8. Having given the equations of two straight lines in space, as

$$\left. \begin{array}{l} x = 3z + 1 \\ y = 2z + 6 \end{array} \right\} \text{ of the 1st ; and } \left\{ \begin{array}{l} x = z + 2 \\ y = -z + \beta' \end{array} \right\} \text{ of the 2d,}$$

to find the value of  $\beta'$ , so that the lines shall actually intersect, and to find the coördinates of the point of intersection.

If the equation

$$\frac{a - a'}{a - a} = \frac{\beta - \beta'}{b - b'}$$

found on page 252, Prop. 4, is satisfied by the constants in the equations of any two straight lines, such lines will intersect. Any five of these constants being given, the equation will determine the sixth. In the present example we have  $x=1$ ,  $x'=2$ ,  $a=3$ ,  $a'=1$ ,  $b=2$ ,  $b'=-1$ ,  $\beta=6$ , from which to find  $\beta'$ .

Placing these values in the above equation it becomes

$$\frac{1 - 2}{3 - 1} = \frac{6 - \beta'}{2 + 1},$$

Whence  $2\beta' = 15$  ;  $\beta' = 7\frac{1}{2}$ . *Ans.*

The formulas for the values of  $x$  and  $y$ , the coördinates of the point of intersection given on the same page, are

$$x = \frac{aa' - aa'}{a - a'}, \quad y = \frac{b\beta' - b\beta'}{b - b'}.$$

Substituting in these the values of  $a$ ,  $a'$ ,  $a$ ,  $a'$ ,  $\beta$ ,  $\beta'$ , we get

$$x = \frac{6-1}{2} = 2\frac{1}{2}, \quad \text{and} \quad y = \frac{15+6}{3} = 7.$$

The value of  $z$  may be found from any one of the given equations, by substituting for  $x$  or  $y$ , the value just found.

Take for example the 1st,  $x=3z+1$ ; it becomes  $2\frac{1}{2}=3z-1$ ; whence  $6z=3$ ,  $z=\frac{1}{2}$ .

Hence the

$$Ans. \begin{cases} \beta^1=7\frac{1}{2}, & y=7. \\ x=2\frac{1}{2}, & z=\frac{1}{2}. \end{cases}$$

Ex. 9. Given the equation of a plane

$$8x=3y+z-4=0,$$

to find the points in which it cuts the three axes, and the perpendicular distance from the origin to the plane.

The point in which the axis of  $X$  pierces the plane is found by making  $y=0$ ,  $z=0$ , in the equation, by which we get  $8x-4=0$ , or  $x=\frac{1}{2}$ . Similarly by making  $x=0$ ,  $z=0$ , we have  $y=-1\frac{1}{3}$ ; and  $x=0$ ,  $y=0$ , gives  $z=4$ .

Formula (7), page 276, which is

$$p = \frac{D}{\sqrt{A^2+B^2+C^2}},$$

will give the perpendicular distance from the origin to the plane by making  $A=8$ ,  $B=-3$ ,  $C=1$ ,  $D=-4$ .

These values give us

$$\begin{aligned} p &= \frac{-4}{\pm\sqrt{64+9+1}} = \frac{4}{\sqrt{74}} = \frac{4\sqrt{74}}{74} \\ &= \frac{\sqrt{1184}}{74} = \frac{34.409}{74} = .4649+. \end{aligned}$$

Hence the

$$Ans. \begin{cases} x_1=\frac{1}{2}, & y_1=-1\frac{1}{3}, & z_1=4. \\ p=.4649+. \end{cases}$$

Ex. 10. Find the equations for the intersections of the two planes,

$$\begin{aligned} 3x-4y+2z-1 &= 0, \\ 7x-3y-z+5 &= 0. \end{aligned}$$

If  $z$  be eliminated between these two equations the resulting equation will be the equation of the projection of the

intersection on the plane ( $xy$ ). We eliminate by multiplying the second equation through by 2 and then adding it to the first, member to member; thus

$$\begin{array}{r} 3x-4y+2z-1=0 \\ \text{Add} \quad 14x-6y-2z+10=0 \\ \hline 17x-10y+9=0. \end{array} \quad \text{1st Ans.}$$

We eliminate  $y$  from the given equations by multiplying the first through by 3, and the second by 4, and subtracting the first result from the second.

$$\begin{array}{r} \text{Thus,} \quad 28x-12y-4z+20=0 \\ \text{Subtract,} \quad 9x-12y+6z-3=0 \\ \hline 19x-10z+23=0. \end{array} \quad \text{2d Ans.}$$

We find the angle included between the two planes by the formula

$$\text{Cos. } V = \frac{AA' + BB' + CC'}{\sqrt{A^2 + B^2 + C^2} \sqrt{A'^2 + B'^2 + C'^2}},$$

in which we must make  $A=3, B=-4, C=2, A'=7, B'=-3, C'=-1$ . These substitutions will give

$$\begin{aligned} \text{Cos. } V &= \frac{21+12-2}{\sqrt{9+16+4} \sqrt{49+9+1}} = \frac{31}{\sqrt{29}\sqrt{59}} = \frac{31}{\sqrt{1711}} \\ &= \frac{31\sqrt{1711}}{1711} = \frac{\sqrt{1644271}}{1711} = \frac{1282.24}{1711}. \end{aligned}$$

$$\text{Hence,} \quad \text{Cos. } V = .74940; \quad V = 41^\circ 27' 41''. \quad \text{Ans.}$$

Ex. 12. The equations of a line in space are

$$x = -2z + 1, \text{ and } y = 3z + 2.$$

Find the inclination of this line to the plane represented by the equation

$$8x - 3y + z - 4 = 0$$

The formula to be used in this example is that found on page 268, Prop. 7, which is

$$\text{Sin. } v = \frac{Aa + Bb + Cc}{\sqrt{1+a^2+b^2} \sqrt{C^2+B^2+A^2}}.$$

In this we must make  $a = -2$ ,  $b = 3$ ,  $A = 8$ ,  $B = -3$ ,  $C = 1$ , which will give

$$\begin{aligned} \text{Sin. } v &= \frac{-16 - 9 + 1}{\sqrt{1+4+9} \sqrt{1+64+9}} = \frac{24}{\sqrt{14} \sqrt{74}} = \frac{12}{\sqrt{7} \sqrt{37}} \\ &= \frac{\sqrt{37296}}{259} = \frac{193.147+}{259} = .74571. \end{aligned}$$

Whence  $v = 48^\circ 13' 13''$ . *Ans.*

**Ex. 13.** Find the angles made by the plane whose equation is

$$8x - 3y + z - 4 = 0$$

with the coördinate planes.

The formulas to be used in this case are

$$\text{Cos. } (xy) = \frac{\pm C}{\sqrt{A^2 + B^2 + C^2}}, \quad \text{cos. } (xz) = \frac{\pm B}{\sqrt{A^2 + B^2 + C^2}},$$

$\text{cos. } (yz) = \frac{\pm A}{\sqrt{A^2 + B^2 + C^2}}$ , and are found on page 264, Prop. 3.

By comparing the given equation with the general equation of the plane, we find,  $A = 8$ ,  $B = -3$ ,  $C = 1$ .

These values being substituted in the first of the above formulas it becomes

$$\text{Cos. } (xy) = \frac{\pm 1}{\sqrt{64 + 9 + 1}} = \frac{\pm 1}{\sqrt{74}} = \frac{1}{\sqrt{74}}$$

$$\text{Cos. } (xy) = \frac{1}{\sqrt{74}} (8.6023) = .11625.$$

The Table gives  $83^\circ 19' 27''$  for the angle whose natural cosine is this decimal.

Similarly the second formula gives

$$\text{Cos. } (xz) = \frac{\mp 3}{\sqrt{74}} = -\frac{3}{\sqrt{74}} = -\frac{3}{\sqrt{74}} = -.34874,$$

and the third

$$\text{Cos. } (yz) = \frac{\pm 8}{\sqrt{74}} = \frac{8}{\sqrt{74}} = \frac{8}{\sqrt{74}} = .92998,$$

which correspond respectively to the angles  $110^\circ 24' 38''$  and  $21^\circ 34' 5''$ .

Hence the

$$\text{Ans. } \begin{cases} 83^\circ 19' 27'' \text{ with the plane } (xy). \\ 110^\circ 24' 38'' \text{ " " " } (xz). \\ 21^\circ 34' 5'' \text{ " " " } (yz). \end{cases}$$

**Ex. 15.** Find the equation of the plane which will cut the axis of  $Z$  at 3, the axis of  $X$  at 4, and the axis of  $Y$  at 5.

The equation of the required plane must have the form

$$Ax + By + Cz + D = 0,$$

and such values of the coefficients  $A$ ,  $B$  and  $C$ , are to be found as will cause the plane represented by the equation to cut the coördinate axes at the specified points.

Referring to the Scholium on page 260, we find the following expressions for the distances from the origin to the points in which a plane cuts the axes, viz. :

$$x = -\frac{D}{A} = OP, \quad y = -\frac{D}{B} = OQ, \quad z = -\frac{D}{C} = OR.$$

If in these we give to  $x$ ,  $y$  and  $z$ , their values by the conditions of the problem, we have

$$3 = -\frac{D}{C}, \quad 5 = -\frac{D}{B}, \quad 4 = -\frac{D}{A}.$$

$$\text{Whence, } C = -\frac{D}{3}, \quad A = -\frac{D}{4}, \quad B = -\frac{D}{5}.$$

Substituting these values in the general equation of the plane it becomes

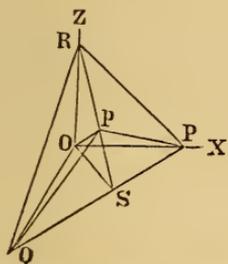
$$-\frac{D}{4}x - \frac{D}{3}y - \frac{D}{3}z + D = 0,$$

$$\text{Or, } 15x + 12y + 20z = 60,$$

by multiplying through by  $-\frac{60}{D}$ ; and by dividing this last equation through by 3 we get finally.

$$\text{Ans. } 5x + 4y + 6\frac{2}{3}z = 20.$$

Ex. 16. Find the equation of the plane which will cut the axis of  $X$  at 3, the axis of  $Z$  at 5, and which will pass at the perpendicular distance 2 from the origin. At what distance from the origin, will this plane cut the axis of  $Y$ ?



Let  $PQR$  be the required plane cutting the axis of  $X$  at  $P$ , making  $OP=3$ , and the axis of  $Z$  at  $R$  making  $OR=5$ .  $Op$  is the perpendicular let fall from the origin upon the plane, and by condition it is equal to 2.

Conceive a plane to be passed through the axis of  $Z$ , and the perpendicular  $Op$  intersecting the required plane in the line  $RS$ , and the plane  $(xy)$  in  $OS$ . Since the plane through  $OR$  and  $Op$  is perpendicular to both the planes  $(xy)$  and  $PQR$ , it is perpendicular to their intersection,  $PQ$ . In the figure we therefore have the right angled triangles  $ROp$ ,  $ROS$ ,  $ROP$ ,  $RPS$ ,  $OPS$  and  $OPQ$ .

From the triangle  $ROp$ , we have

$$Rp = \sqrt{RO^2 - Op^2} = \sqrt{5^2 - 2^2} = \sqrt{21},$$

and from the triangles  $ROp$ ,  $ROS$ ,

$$Rp : OR :: OR : RS,$$

$$\text{Or,} \quad \sqrt{21} : 5 :: 5 : RS = \frac{25}{\sqrt{21}}.$$

The triangle  $ROP$  gives

$$RP = \sqrt{RO^2 + PO^2} = \sqrt{5^2 + 3^2} = \sqrt{34},$$

$$\text{Whence,} \quad PS = \sqrt{RP^2 - RS^2} = \sqrt{34 - \frac{625}{21}} = \sqrt{\frac{39}{21}}.$$

Then from the similar right angled triangles  $OPS$ ,  $OPQ$ , we have

$$PS : OP :: OP : PQ$$

$$\text{Or, } \sqrt{\frac{9}{21}} : 3 :: 3 : PQ = \frac{9}{\sqrt{\frac{9}{21}}} = \frac{9\sqrt{21}}{\sqrt{89}}.$$

$$\text{Whence, } OQ = \sqrt{PQ^2 - OP^2} = \sqrt{\frac{81 \times 21}{89} - 9} = \sqrt{\frac{900}{89}} \\ = \frac{30}{\sqrt{89}}, \text{ which may have either sign.}$$

The equation of the plane will be what the equation

$$Ax + By + Cz + D = 0$$

becomes, when we substitute in it the values of  $A$ ,  $B$  and  $C$ , given by the equations

$$5 = -\frac{D}{C}, \quad 3 = -\frac{D}{A}, \quad \frac{30}{\sqrt{89}} = -\frac{D}{B}.$$

We thus get

$$-\frac{D}{3}x - \frac{\sqrt{89}D}{30}y - \frac{D}{5}z + D = 0,$$

$$\text{Or, } 10x + \sqrt{89}y + 6z - 30 = 0,$$

which is the equation of the required plane, and it cuts the axis of  $Y$  at the distance  $OQ = \frac{30}{\sqrt{89}}$ , from the origin.

**Ex. 17.** Find the equations of the intersection of the two planes whose equations are

$$3x - 2y - z - 4 = 0,$$

$$7x + 3y + z - 2 = 0.$$

By adding these equations, member to member, we eliminate  $z$  and get for our result

$$10x + y - 6 = 0$$

which is the equation of the projection of the intersection of the two planes on the plane  $(xy)$ .

Multiplying the same equations through, the first by 3, and the second by 2, we have

$$9x - 6y - 3z - 12 = 0,$$

Add,  $14x + 6y + 2z - 4 = 0$

And our result  $23x - z - 16 = 0$ , is the equation of the projection of the intersection on the plane ( $xz$ ).

In like manner multiplying the first through by 7, and the second by 3, we find

$$21x - 14y - 7z - 28 = 0,$$

Subtract,  $21x + 9y + 3z - 6 = 0,$

And our result,  $23y + 10z + 22 = 0$ , is the equation of the projection of the intersection on the plane ( $yz$ ).

Ex. 18. Find the inclination of the planes whose equations are expressed in Example 17.

The equations of the planes are

$$3x - 2y - z - 4 = 0,$$

$$7x + 3y + z - 2 = 0.$$

The formula for the cosine of the inclination of these two planes is

$$\text{Cos. } V = \frac{AA' + BB' + CC'}{\sqrt{A^2 + B^2 + C^2} \sqrt{A'^2 + B'^2 + C'^2}},$$

found on page 267, Prop 6. In this we must make the following substitutions, viz. :

$$A = 3, B = -2, C = -1; A' = 7, B' = 3, C' = +1,$$

by which we get

$$\text{Cos. } V = \frac{21 - 6 - 1}{\sqrt{9 + 4 + 1} \sqrt{49 + 9 + 1}} = \frac{14}{\sqrt{14} \sqrt{59}} = \frac{\sqrt{14}}{\sqrt{59}},$$

$$\text{Whence, } \text{Cos. } V = \frac{\sqrt{14}}{\sqrt{59}} = \frac{\sqrt{14} \sqrt{59}}{59} = \frac{\sqrt{826}}{59} = .48712,$$

and by the Table this is found to correspond to the angle  $60^\circ 50' 55''$ , or the supplementary angle,  $119^\circ 9' 5''$ . *Ans.*

Ex. 19. A plane intersects the coördinate plane ( $xz$ ), at an inclination of  $50^\circ$ , and the coördinate plane ( $yz$ ), at an inclination of  $84^\circ$ . At what angle will this plane intersect the plane ( $xy$ )?

We employ in this case the formula

$$\text{Cos.}^2(xy) + \text{cos.}^2(xz) + \text{cos.}^2(yz) = 1,$$

which is found on page 264. From it we get

$$\text{Cos.}^2(xy) = 1 - \text{cos.}^2(xz) - \text{cos.}^2(yz).$$

But  $\text{cos.}(xz) = \text{cos.}50^\circ = .64279$ , and  $\text{cos.}(yz) = \text{cos.}84^\circ = .10453$ .

Whence,  $\text{Cos.}^2(xy) = 1 - .4241054050 - .0109265209$

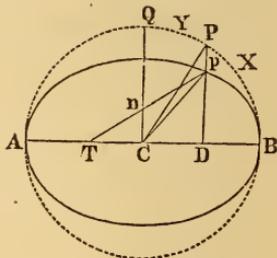
$$\text{Cos.}(xy) = \sqrt{.5758945950} = .75887,$$

which by the Table corresponds to the angle  $40^\circ 38' 6''$ . *Ans.*

### MISCELLANEOUS PROBLEMS.

Ex. 1. The greatest or major axis of an ellipse is 40 feet, and a line drawn from the center, making an angle of  $36^\circ$  with the major axis and terminating in the ellipse, is 18 feet long; required the minor axis of this ellipse, its area and eccentricity.

Suppose  $AB$  to be the major axis,  $C$  the center, and  $F$  the focus of the ellipse, and let  $Cp$  be the line drawn from the center making an angle of  $36^\circ$  with  $AB$ , and terminating in the curve at  $p$ . From  $p$  let fall the perpendicular  $pD$  on the axis, and produce this perpendicular to meet the circumference described on  $AB$ , as a diameter in  $P$ , and draw  $CP$ .



Then in the right angled triangle  $CpD$  we have

$$1 : \sin. pCD :: Cp : pD = Cp.\sin. pCD.$$

$$1 : \cos. pCD :: Cp : CD = Cp.\cos. pCD.$$

In the Table we find  $\sin. 36^\circ = .58779$ ,  $\cos. 36^\circ = .80902$ ,

Hence,  $pD = 18 \times .58779 = 10.58022$ ,

And  $CD = 18 \times .80902 = 14.56236$ .

From the right angled triangle  $CPD$ , we get

$$\begin{aligned} PD &= \sqrt{CP^2 - CD^2} = \sqrt{400 - 212.062439} \\ &= \sqrt{187.937561} = 13.709. \end{aligned}$$

Now we have an ordinate of the ellipse drawn to its transverse axis, and the corresponding ordinate of the circle described on that axis. Calling the semi-transverse axis  $A$ , and the semi-conjugate axis  $B$ , we have,

$$A : B :: PD : pD,$$

That is  $20 : B :: 13.709 : 10.58022$ .

Whence,  $B = \frac{20 \times 10.58022}{13.709} = 15.4376$ ,

And  $2B = 30.8752$ .

The area of an ellipse is measured by the product of its semi-axes multiplied by 3.14159+.

Hence for the required ellipse, we have

$$\text{Area} = \pi AB = 3.14159 \times 20 \times 15.4376 = 969.972+.$$

Denoting the eccentricity by  $E$ , we have

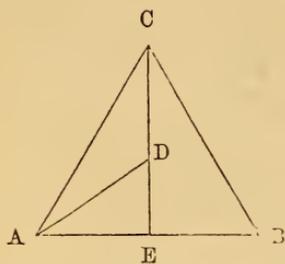
$$E = \frac{\sqrt{A^2 - B^2}}{A} = \frac{\sqrt{400 - 238.3195}}{20} = \frac{12.715}{20} = .63575.$$

Collecting our results we therefore have

$$\text{Ans. } \left\{ \begin{array}{l} \text{Minor axis, } 30.8752 \text{ feet.} \\ \text{Area, } 969.972 \text{ sq. feet.} \\ \text{Eccentricity, } .63575 \text{ feet.} \end{array} \right.$$

In the second miscellaneous problem it was assumed that if the side of an equilateral triangle be denoted by  $a$ , the

line drawn from the center to the vertex of either angle of the triangle, will be represented by  $\frac{a}{\sqrt{3}}$ . To prove this, let



$ABC$  be an equilateral triangle of which  $D$  is the center, by which we mean the center of the inscribed or circumscribed circle. Draw  $AD$  and  $CD$ , producing the latter to meet the base of the triangle in  $E$ .

Now since the angle  $CAE$  is  $60^\circ$  and  $AD$  bisects this angle, the angle  $DAE$  is  $30^\circ$ , and  $DE$  is therefore equal to one half of  $AD$ . Denote  $AB$  by  $a$  and  $DE$  by  $x$ ; then  $AD = 2x$ .

In the right angled triangle  $ADE$ , we have

$$\overline{AE}^2 + \overline{DE}^2 = \overline{AD}^2,$$

$$\text{Or,} \quad \frac{a^2}{4} + x^2 = 4x^2, \quad 3x^2 = \frac{a^2}{4}.$$

$$\text{Hence,} \quad x = \frac{a}{2\sqrt{3}}, \quad 2x = AD = \frac{a}{\sqrt{3}},$$

which was to be proved.

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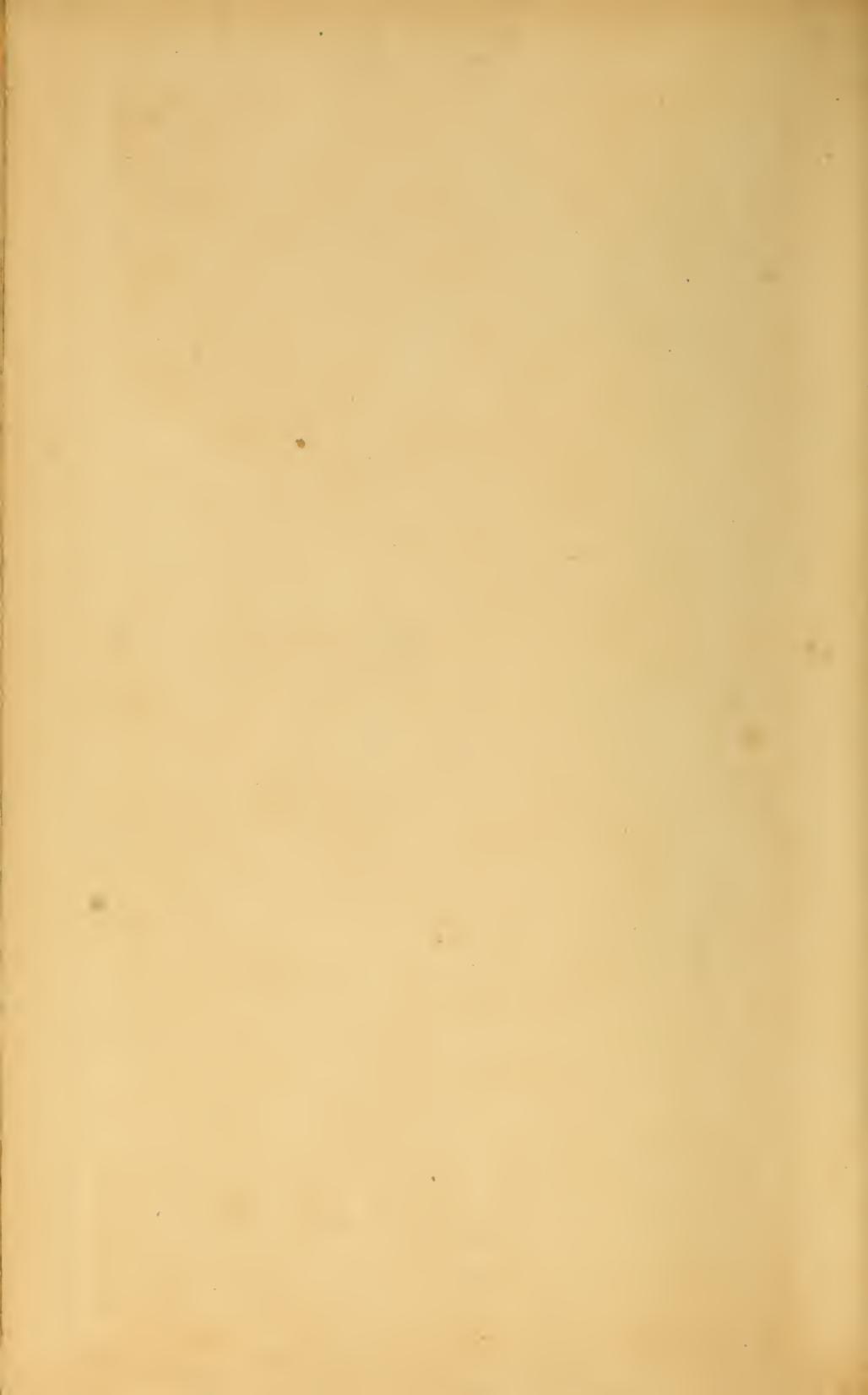
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